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AFAL TR-75-118

SYSTEMATIC DESIGN OF MODULAR ESTIMATORS FOR AIRCRAFT NAVIGATION

Julian L. Center

THE ANALYTIC SCIENCES CORPORATION

Technical Report AFAL-TR-75-118

July 1975

Air Force Avionics Laboratory
Air Force Systems Command
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Ohio 45433

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Recent hardware advances have made it practical to build relatively sophisticated data preprocessors into sensor packages. To take full advantage of these developments, a navigation system could use an estimation architecture with data preprocessors located in the sensor packages and a data coordinator located with the information user. This type of data processor, termed herein a "modular estimator," has a structure which makes possible:

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- parallel computation
- modular design
- interface standardization
- system reliability improvement
- channel capacity reduction

This report presents a systematic procedure for designing modular estimators and applies the procedure to a simple example. The mathematical foundations for this procedure, including new rate-distortion theory results, are presented in Appendix A.

FOREWORD

This report, AFAL-TR-75-118, was submitted on 30 April 1975 as TR-316-4-3 by The Analytic Sciences Corporation, 6 Jacob Way, Reading, Massachusetts 01867, under contract F33615-72-C-1787, Advanced Development Program 666A, Task Number 02, with the Air Force Avionics Laboratory. This document covers research performed during the period January 1974 through April 1975. The author acknowledges the assistance of the Air Force Project Engineer, Captain Harvey Brock.

This technical report has been reviewed and is approved for publication.

FOR THE DIRECTOR

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Delivery Division

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1. INTRODUCTION

1.1 MOTIVATION

A hybrid aircraft navigation system processes data from a variety of sensors to produce estimates of the aircraft's attitude, position, and velocity. These estimates of the aircraft's state are displayed to the pilot and used by the aircraft's automatic control systems. A typical hybrid navigation system is illustrated in Fig. 1-1.

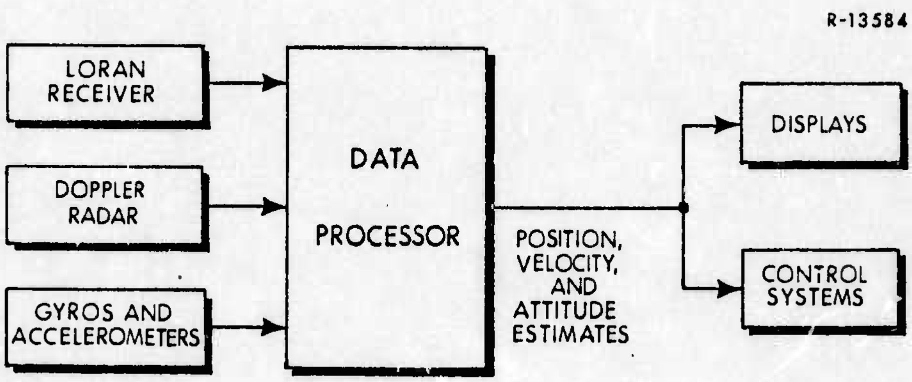

Figure 1-1 Typical Hybrid Navigation System Structure

Theoretically, the most accurate estimates of the system states would be produced by a single data processor operating on all the raw sensor data simultaneously. However, such a centralized estimation architecture requires communication of large quantities of information to a central computer. Furthermore, an extremely fast central computer is needed to process all of the raw data at sensor data rates. Since the cost of a communications channel increases with bandwidth and the cost of a computer increases with speed, a centralized estimation system can become quite costly. Also a centralized estimation architecture is not conducive

to a modular system design since a change in one of the sensors can impact the mechanical and electical design of the entire system.

Fortunately, an attractive alternative to the centralized estimation architecture is now available. Recent advances in large scale integration (LSI) electronics have made data processing units available which are physically small, lightweight, cheap, and low in power consumption. Thus it is feasible to consider modular estimation architectures which perform relatively sophisticated data preprocessing within each sensor package, as illustrated in Fig. 1-2. In this modular estimation architecture, data preprocessors are operated at sensor data rates to remove redundancy in the sensor data. The resulting compressed information is transmitted to a data coordinator, which supplies estimates at a slower rate to the displays and controls. This redundancy reduction allows a relaxation of the bandwidth requirements for communication channels which transmit data from the sensors. Furthermore, since computations are being performed in parallel in this system, the speed and capacity of any one computer in the system can be much less than that required for the computer in the centralized estimation architecture.

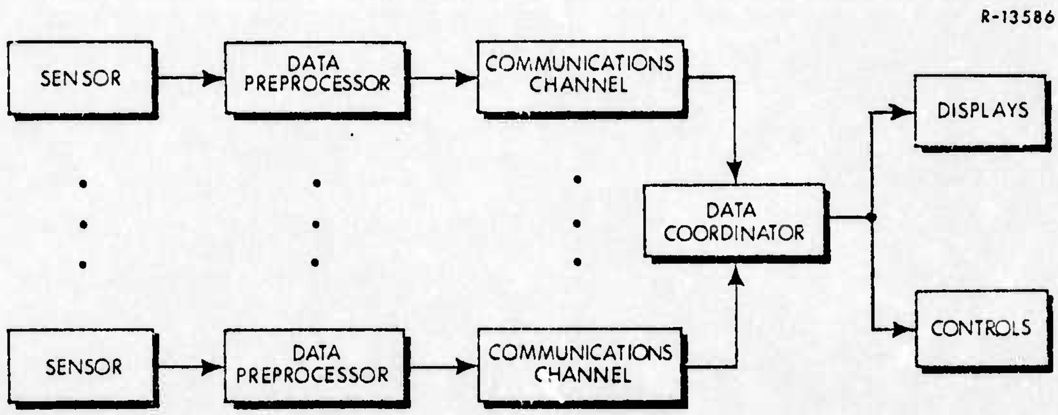

Figure 1-2 Modular Estimation Architecture

A major dividend of the modular architecture results because electrical and mechanical interfaces between the data coordinator and certain functional types of sensor/preprocessor package can be standardized. Then any velocity reference can be replaced by another velocity reference without affecting the electrical design of the system. Also, the data preprocessors can be designed so that a failure of any one component, even part of the data coordinator, will still leave the system operational. This increases the system reliability.

Actually, state-of-the-art navigation systems already use a form of modular estimation. A typical state-of-the-art navigation system, diagrammed in Fig. 1-3, preprocesses accelerometer and gyro measurements by implementing a set of equations, called mechanization equations, to produce position, velocity, and attitude estimates. Signals from radio navigation aids are also preprocessed by signal detection and demodulation techniques, which serve to compress the data so that information not required for navigation is removed and relevant information is retained. Although these preprocessing equations are generally nonlinear, the errors

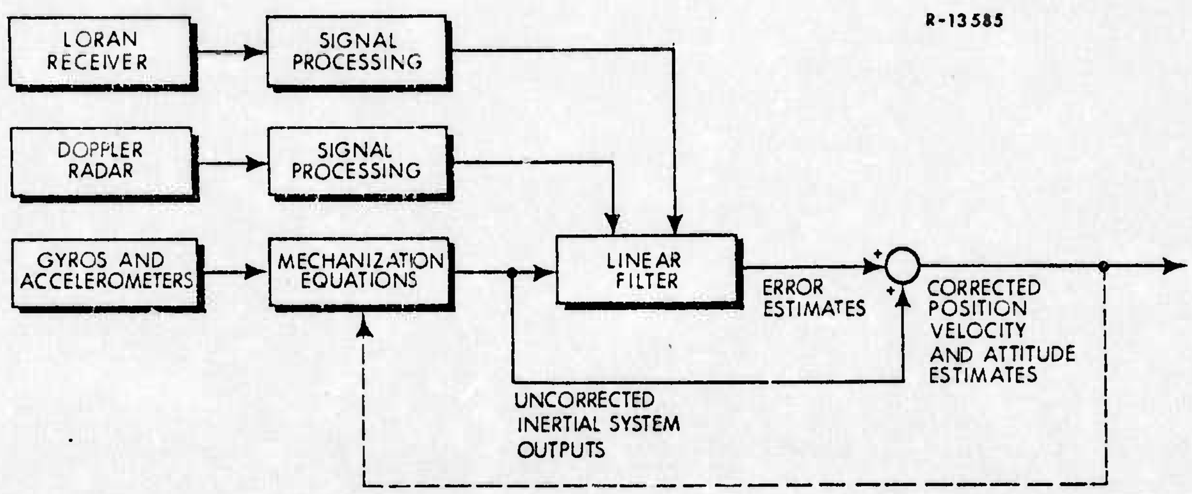

Figure 1-3 Typical State-of-the-Art Navigation System

in the resulting compressed data can usually be modeled as the outputs of a linear system driven by Gaussian noise. Therefore, a linear filter can be used to coordinate the compressed sensor data and determine the minimum-variance estimates of the navigation system errors. These error estimates are then used to correct the inertial system outputs, and the corrected output may be used to reset the mechanization equations.

This report presents original work relating to the design of modular estimators. The mathematical foundations of a systematic procedure for designing modular estimators are established and the procedure is applied to a simple example.

1.2 DESIGN CONSIDERATIONS

In designing a modular estimator, the following factors must be balanced:

- estimation accuracy
- communication channel bandwidths
- data preprocessor and data coordinator complexity

Estimation accuracy is usually measured in terms of the mean-square estimation error for some portion of the system state. In information theory terminology, this is referred to as a quadratic distortion measure. A quadratic distortion measure will be assumed almost exclusively in this report.

The bandwidth of a communications channel is also called its channel capacity or information transfer rate and is usually measured in terms of the number of bits per second it is capable of handling. The cost of a communications channel increases as its capacity is increased; therefore, it is desirable to keep the required channel capacity small. On the other hand, transmitting the sensor data over a limited capacity channel introduces distortion (e.g., measurement quantization increases mean-square estimation-error). The distortion introduced depends on the coding used to convert the data to a form which can be transmitted over the channel. A particular coding scheme applied to data from a particular source will achieve a specific information transfer rate (bits transferred per second) and distortion (e.g. mean-square estimation-error) corresponding to a point in the rate-distortion plane. As an example, consider the problem of sending the values of Gaussian random variables over a digital channel. Suppose one value must be sent every second. A straight-forward way of coding the value is by quantization. A wide variety of quantization methods, which differ in both the number and location of quantization steps, can be used. The number of quantization steps determines the number of bits sent every second, and the location of the steps determines the mean-square error in the transmitted value.

For a particular data source, the relationship between the channel capacity used and the minimum distortion which can be achieved is given by the rate-distortion curve (also called the rate-distortion function) as shown in Fig. 1.2-1. The rate-distortion curve is the lower left boundary of the region of points corresponding to various coding systems. Both the required channel capacity and the distortion can be minimized by chosing a coding scheme which

gives performance near this curve. Analytical methods, discussed in Appendix A, can be used to determine the rate-distortion curve, but in general the coding scheme that will give performance on this curve when combined with a specific channel cannot be determined analytically.

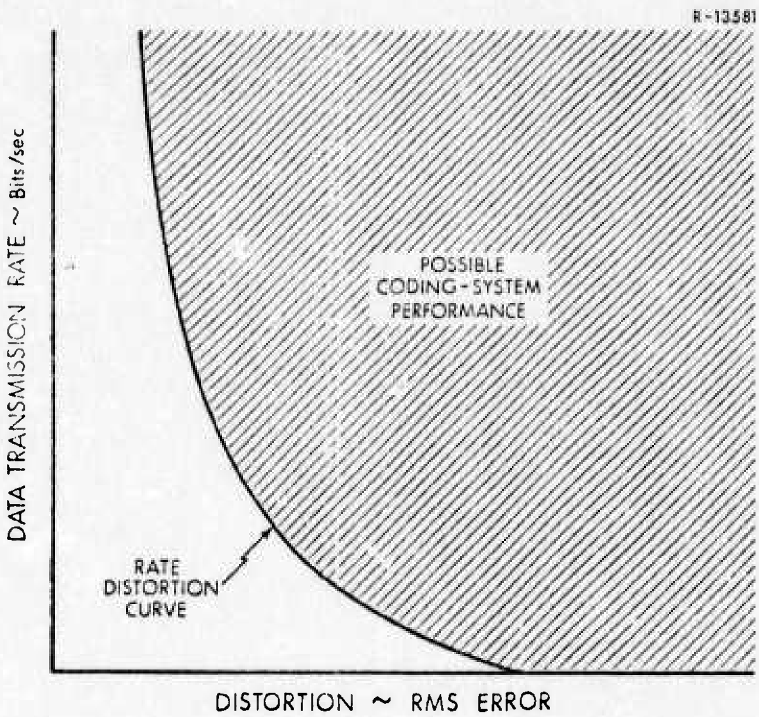

Figure 1.2-1 Typical Rate-Distortion Curve

Performance near the rate-distortion curve can be achieved by using a data preprocessor to remove irrelevant and redundant information from the sensor data. The complexity of the preprocessor must be held to a minimum to minimize costs. Similarly, the data coordinator, which combines data transmitted from the data preprocessors, must be kept as simple as possible to keep computer costs down. Unfortunately, simplifying the data preprocessor increases the required channel capacity and simplifying the data

coordinator increases the distortion. So a design compromise must be found.

The methods discussed in this report provide a systematic procedure for striking a balance among the conflicting goals of minimizing channel capacity requirements, minimizing distortion, and minimizing estimator complexity. The next section presents an overview of this procedure.

1.3 OVERVIEW OF DESIGN PROCEDURE

The modular estimator design technique presented here is composed of two major steps: preprocessor design and data coordinator design. Preprocessor design studies determine a feasible combination of prefilter structure, times of transmission, channel capacity and coding scheme. Data coordinator design studies are used to develop a practical data coordinator that will efficiently use all available data.

Data preprocessors are designed using the procedure described in Chapter 2. The minimum-variance reduced-order estimator design techniques of Ref. 7 are used to study various prefilter structures to determine the distortion limit that can be achieved with a very high capacity channel. Next the techniques developed in Appendix A are used to compute the rate-distortion curve for a selection of candidate prefilters. The times of transmitting data from the data preprocessor to the data coordinator also affect the rate-distortion curves; so, candidate prefilters are studied with a variety of transmission times. As a result of these studies, a feasible combination of prefilter, transmission times, and channel capacity is determined. A coding scheme that approximates the optimal coding scheme is also determined at this time.

Once feasible preprocessor designs are established, the data coordinator is designed as described in Chapter 3. Preprocessor dynamics as well as system dynamics are considered in this design, but the data coordinator is usually designed with fewer than the number of states required to achieve optimal performance. The minimum-variance reduced-order estimator design procedures of Ref. 7 are used effectively in data coordinator design. The sensitivity of a data coordinator design to uncertainties in system parameters is studies by applying the methods discussed in Chapter 4 and Appendix B.

2. DATA PREPROCESSOR DESIGN

2.1 PREPROCESSOR STRUCTURES

A data preprocessor is a device or algorithm which transforms sensor data into an easily used form, removes irrelevant and redundant information, and encodes the data for transmission to the user. Therefore, preprocessor operation can be divided into three basic functions (See Fig. 2.1-1):

- Data Transformation - converting raw sensor data to system state measurements, e.g., a doppler receiver converts electromagnetic signals to a velocity measurement. Usually this is a nonlinear operation.
- Prefiltering (Data Compression) - reducing the volume of data which must be communicated to the data coordinator, e.g., doppler velocity data may be averaged before being sent to the data coordinator. Usually this is a linear operation.
- Encoding - coding the prefiltered data in a form suitable for transmission over the channel. For example, velocity data may be quantized for transmission over a digital channel. Usually this is a nonlinear operation.

For convenience, the data transformation will be ignored here. Equivalently, the data transformation can be thought of as part of the sensor.

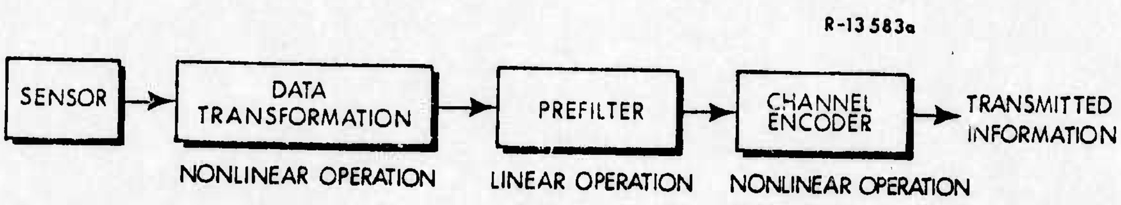

Figure 2.1-1 Typical Data Preprocessor Structure

Although nonlinear operations are sometimes required to transform raw sensor data to system state measurements, linear operations are usually adequate for prefiltering. For example, simple averaging of polynomial fits to the data are often used for prefiltering (Refs. 1 through 3). Therefore, only linear prefiltering techniques will be considered here. Furthermore only discrete-time prefiltering which can be performed with digital hardware will be considered.

To simplify the system design, the following groundrules will be used:

- Prefilters will be designed to operate independently of other information sources.
- The information transfer rate will be assumed to be fixed at a specified number of bits per transmission.
- A quadratic distortion measure will be used to measure the performance of the prefilter.

The first groundrule limits the number of interconnections between system elements to the form shown in Fig. 1-2. Furthermore, it leads to a design which can give acceptable performance when some of the sensors fail. The other groundrules reflect common engineering practice and do not greatly restrict the design.

2.2 OPTIMUM PREPROCESSOR DESIGN

Suppose that the signal process produced by the sensor (possibly after a nonlinear data transformation) can

be modeled as the output of a linear system driven by Gaussian noise and that the distortion measure is quadratic. Then results from Refs. 4 and 5 and from Appendix A show that the optimum preprocessor takes the form shown in Fig. 2.2-1. The raw sensor data is fed into the Kalman filter for this signal source, and the state of this Kalman filter is encoded for transmission to the data coordinator.

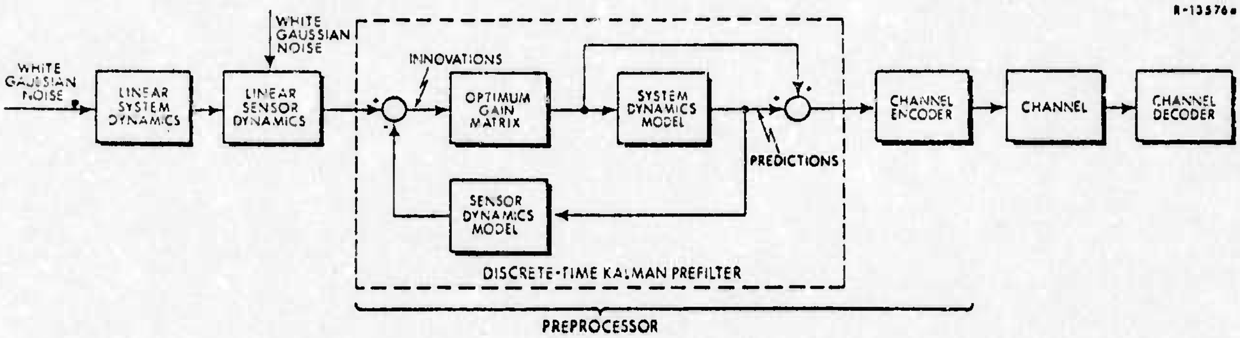

Figure 2.2-1 Optimum Preprocessor for Gaussian Signals

Since the state of the Kalman filter is a continuous random variable, an infinite number of bits would be required to transmit its exact value. Because the encoder, channel, and decoder can handle only a finite number of bits per second, distortion will be introduced in transmitting the state of the prefilter to the data coordinator. The minimum distortion which can be achieved with a channel of specified capacity is given by the rate-distortion curve (see Appendix A). Also, the rate-distortion curve determines the minimum channel capacity needed to achieve a specified distortion. Thus, <u>the rate-distortion curve permits the separation of prefilter design and encoder-channel-decoder design</u>. Figure 2.2-2 shows a typical rate-distortion curve.

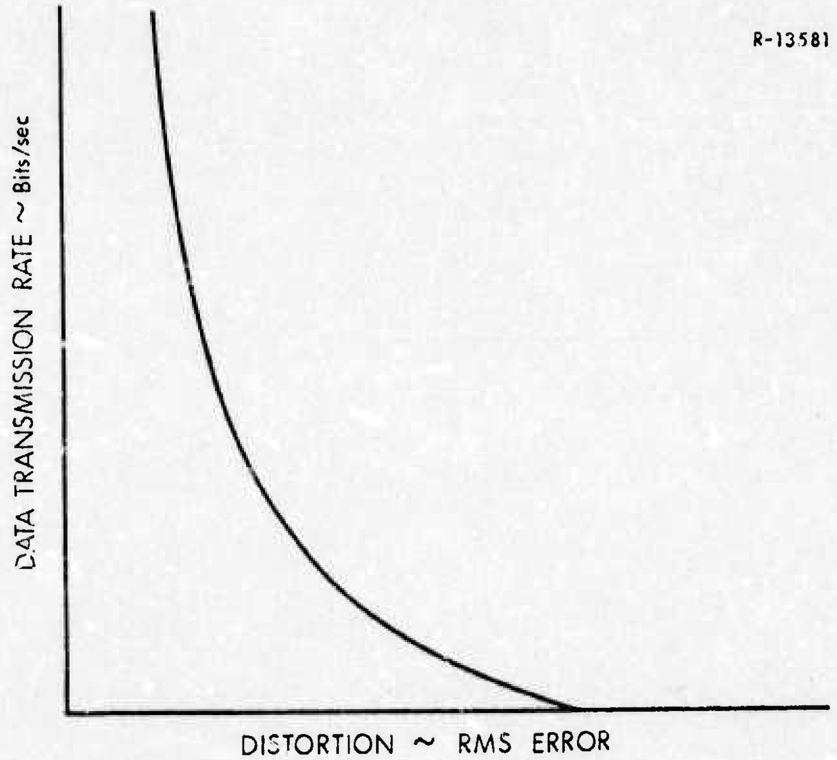

Figure 2.2-2 Typical Rate-Distortion Curve

The rate-distortion curve specifies the channel capacity needed to achieve a desired level of distortion provided that an optimum prefilter (Kalman filter) and optimum encoder-channel-decoder combination are used. During the calculation of the rate-distortion function, the optimum combined effect of the encoder, channel, and decoder is determined. The optimum combination has the appearance of an encoder-compression matrix multiplying the filter state, an additive white Gaussian noise, introduced by the channel, and another Kalman filter used as a decoder (see Fig. 2.2-3). The rate-distortion calculations in Appendix A determine the optimum encoder compression matrix to

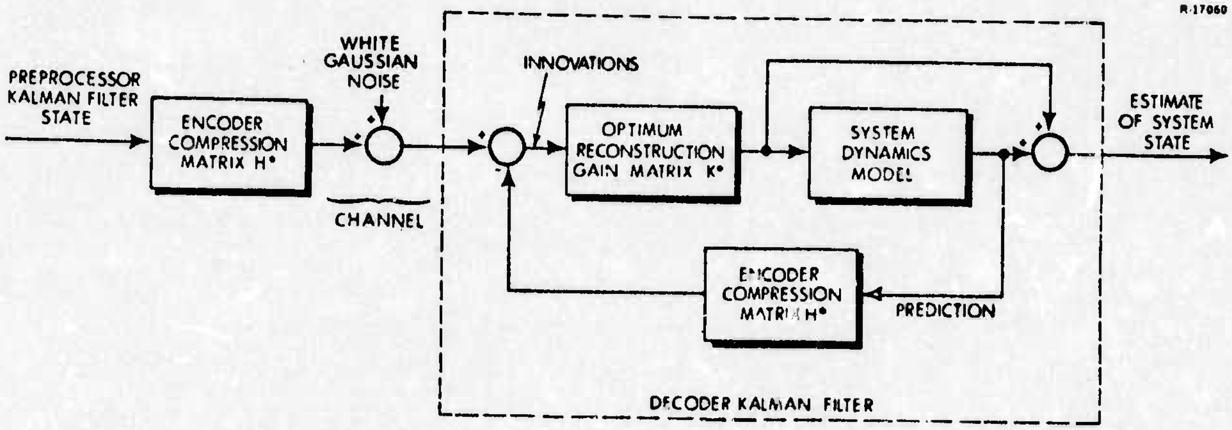

Figure 2.2-3 Optimum Encoder-Channel-Decoder Combination

use with a given channel noise. In effect, this determines the optimum signal-to-noise ratio and the most important part of the system state to transmit.

In most situations, the channel cannot be chosen to be simply an additive Gaussian noise with a specified covariance matrix. For example, digital communications channels produce quantization errors which are neither additive nor Gaussian. In theory, an encoder and a corresponding decoder exist, which when combined with the digital channel, would be equivalent to the optimum combination shown in Fig. 2.2-3. Unfortunately, the structure of the optimum encoder and decoder for a digital channel is not determined during the calculation of the rate-distortion curve. However, the differential pulse code modulation (DPCM) system shown in Fig. 2.2-4 can achieve performance close to the theoretical bound (Ref. 6). The DPCM encoder removes information which can be predicted by the decoder using previously transmitted information and a system dynamics model. The same dynamics model is also used at the receiver to reconstruct

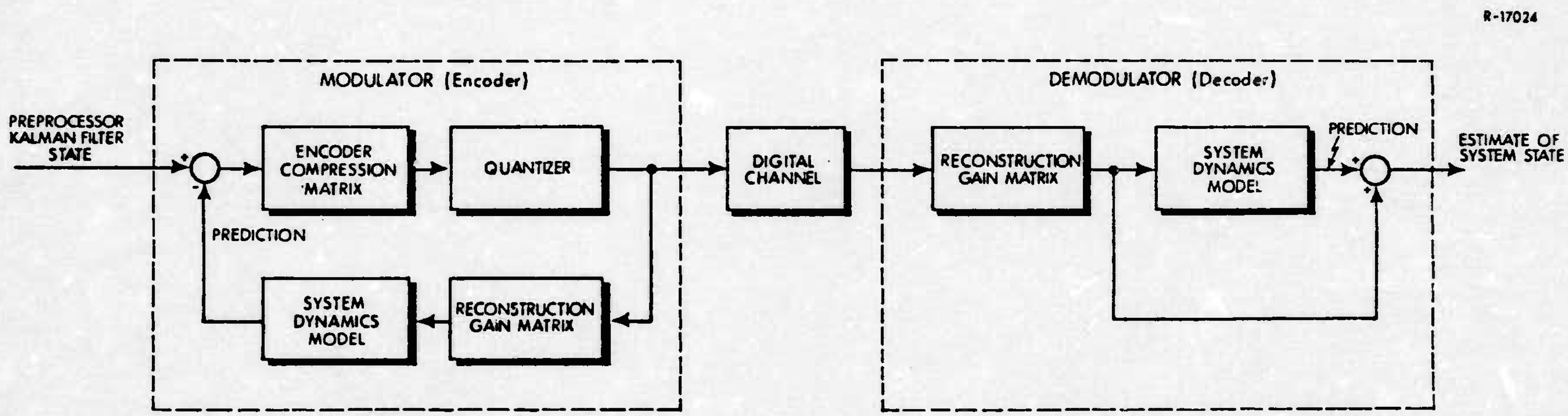

Figure 2.2-4 Differential Pulse Code Modulation-Demodulation System

the original signal. The reconstruction gain used in both the encoder and decoder is similar to the optimum gain in a Kalman filter.

The complexity of the Kalman filter used for the optimum data prefilter often precludes its implementation. Therefore, preprocessor designs which approximate the optimum but are less complex are discussed in the next section. Since the use of suboptimal prefilters introduces additional distortion in the data transmission, techniques for quantifying this performance degradation are also discussed.

2.3 SUBOPTIMAL PREPROCESSOR DESIGN

Instead of estimating the entire state, the data prefilter can be designed to estimate only a portion of the sensor and system state vector. This allows a reduction in the complexity of the preprocessor. In this case, $\underline{y}(t_n)$ is designed to be an estimate of $S(t_n)\ \underline{x}(t_n)$, where $\underline{x}(t_n)$ is the system state vector at time t_n and $S(t_n)$ is a matrix which selects linear combinations of the states for estimation. For example, it is reasonable to choose the output of the prefilter associated with a doppler radar receiver to be an estimate of aircraft velocity.

One of the simplest prefilters merely averages the raw data. This type of prefilter is most useful when the signal can be assumed to be essentially constant between data transmission times and when measurement errors can be assumed to be white noise. When these assumptions are not justified, more sophisticated prefiltering may be required.

The minimum-variance reduced-order (MVRO) estimator techniques of Ref. 7 provide tools for designing

prefilters which minimize the variance of the estimation error subject to constraints on the complexity of the estimator. Assume that the state variable representation of the system is

$$\underline{x}(t_{n+1}) = \Phi(t_n)\, \underline{x}(t_n) + \underline{w}(t_n) \tag{2.3-1}$$

and the measurements taken between t_n and t_{n+1} can be represented by

$$\underline{z}(t_n) = H(t_n)\, \underline{x}(t_n) + \underline{v}(t_n) \tag{2.3-2}$$

Then a MVRO estimator for the system is described by the equations

$$\begin{aligned} \underline{m}(t_{n+1}) = \; & T(t_{n+1})\, \Phi(t_n)\, C(t_n)\, \underline{m}(t_n) \\ & + T(t_{n+1})\, \Phi(t_n)\, K(t_n)\, \underline{\tilde{z}}(t_n) \end{aligned} \tag{2.3-3}$$

$$\begin{aligned} \underline{y}(t_n) = \; & S(t_n)\, C(t_n)\, \underline{m}(t_n) \\ & + S(t_n)\, K(t_n)\, \underline{\tilde{z}}(t_n) \end{aligned} \tag{2.3-4}$$

where

$\underline{z}(t_n)$ = the raw sensor measurement at time t_n

$\underline{m}(t_n)$ = a memory of measurements prior to t_n

$\underline{y}(t_n)$ = the compressed data vector at time t_n

$\underline{\tilde{z}}(t_n)$ = $\underline{z}(t_n) - H(t_n)\, C(t_n)\, \underline{m}(t_n)$ (2.3-5)

= the innovations for the reduced-order estimator

and where $T(t_{n+1})$ is a matrix of design parameters which selects which portion of the state the memory vector, $\underline{m}(t_{n+1})$ will estimate. The matrices $C(t_n)$ and $K(t_n)$ are determined completely by $T(t_n)$ and the system parameters. The innovations sequence is the new information supplied to the

reduced-order estimator by the measurements. The innovation sequence for a Kalman filter is white, but the innovation sequence for a reduced-order estimator is correlated.

A diagram of a MVRO estimator for the system is shown in Fig. 2.3-1. Unfortunately, the computations required to determine the optimum parameters $C(t_n)$ and $K(t_n)$ are quite complex. If the application permits these parameters to be precomputed and stored, then the MVRO estimator can be used as the prefilter. When this is not possible, heuristic prefilter designs must be used. However, even in this case, the MVRO estimator design procedures can provide guidelines for prefilter design and a measure of how well a prefilter with a given state selection can be expected to perform.

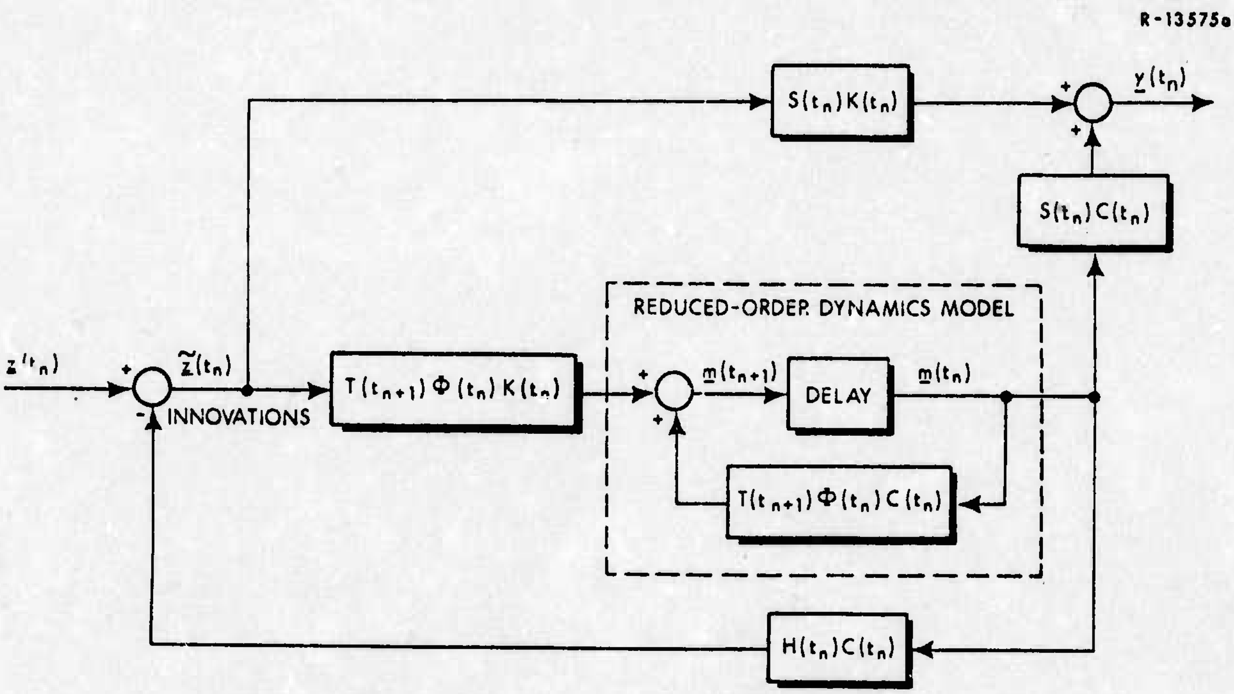

Figure 2.3-1 MVRO Prefilter

In general, the data prefilter can be described by the equations

$$\underline{m}(t_{n+1}) = A(t_n)\, \underline{m}(t_n) + B(t_n)\, \underline{\tilde{z}}(t_n) \qquad (2.3\text{-}6)$$

$$\begin{aligned} \underline{y}(t_n) = {} & S(t_n)\, C(t_n)\, \underline{m}(t_n) \\ & + S(t_n)\, K(t_n)\, \underline{\tilde{z}}(t_n) \end{aligned} \qquad (2.3\text{-}7)$$

$$\underline{\tilde{z}}(t_n) = \underline{z}(t_n) - G(t_n)\, \underline{m}(t_n) \qquad (2.3\text{-}8)$$

Figure 2.3-2 shows the block diagram for this representation of the data prefilter. Note that the MVRO prefilter has this structure with

$$A(t_n) = T(t_{n+1})\, \Phi(t_n)\, C(t_n) \qquad (2.3\text{-}9)$$

$$B(t_n) = T(t_{n+1})\, \Phi(t_n)\, K(t_n) \qquad (2.3\text{-}10)$$

$$G(t_n) = H(t_n)\, C(t_n)$$

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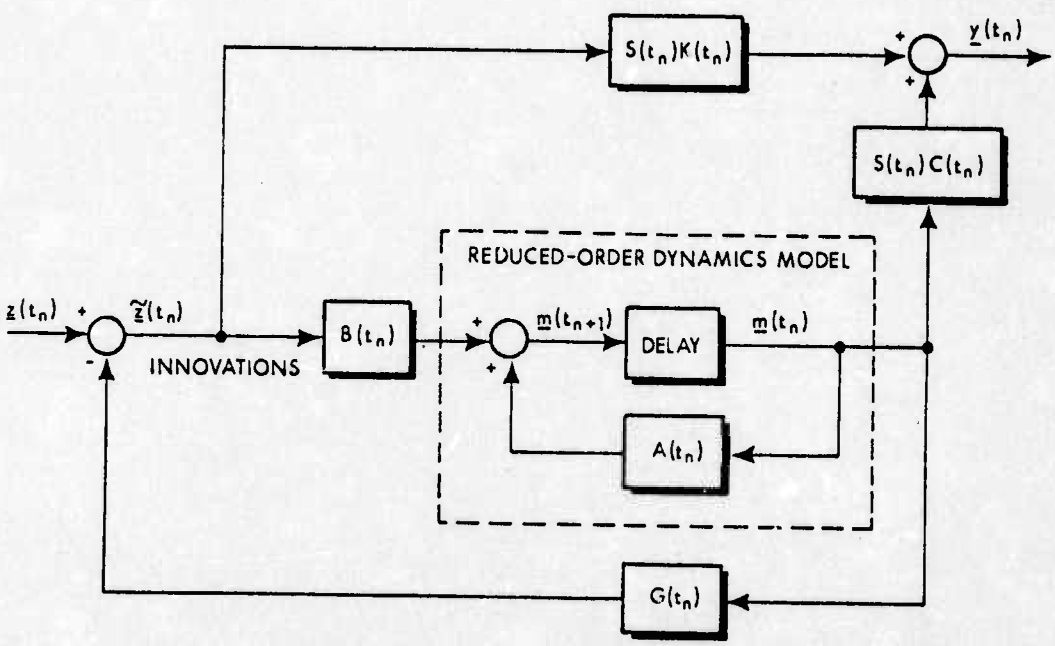

Figure 2.3-2 General Prefilter Structure

When a suboptimal prefilter is used, the optimal encoder-channel-decoder combination can be determined by considering the prefilter as part of the sensor dynamics. The optimum decoder is the Kalman filter which models both the system and prefilter dynamics. The rate-distortion functions for a selection of candidate prefilters can be determined using the methods of Appendix A. The prefilter whose rate-distortion curve lies closest to the "system with sensor" (optimal prefilter) rate-distortion curve at the desired transmission rate is judged to be best (see Fig. 2.3-3). In some cases, the optimum encoder-channel-decoder is not practical to implement because it involves building a Kalman filter which models both the system <u>and</u> prefilter dynamics. In these situations either the MVRO estimator design techniques of Ref. 7 or the following heuristic procedure can be used to determine an encoder-channel-decoder combination which is practical to implement:

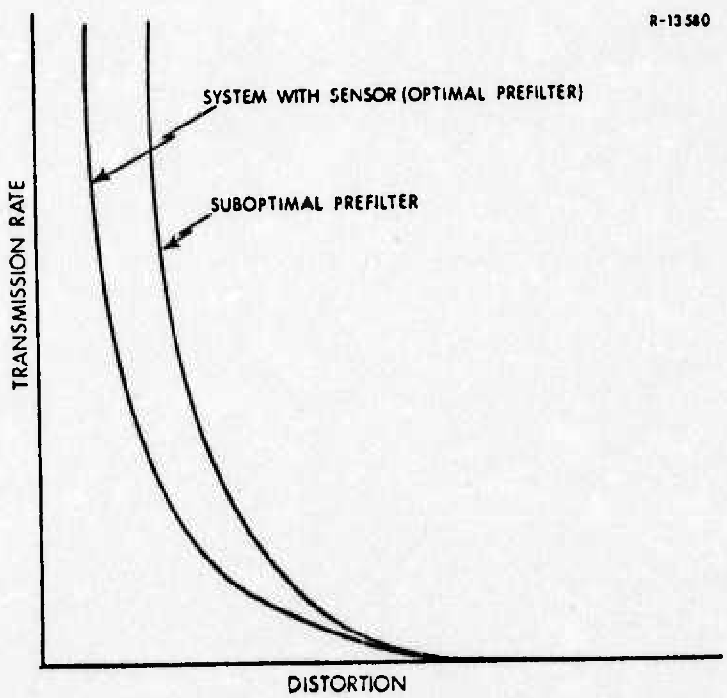

Figure 2.3-3 Typical Rate-Distortion Functions

(1) The prefilter is put in the form of Fig. 2.3-2. This can be done for most linear prefilters (not just MVRO prefilters).

(2) The innovations sequence is replaced by a white noise sequence with the same variance. (The innovations sequence for a Kalman filter is already white.)

(3) The methods of Appendix A are used to compute the encoder compression matrix H* and the reconstruction gain matrix K*.

This encoder-channel-decoder design procedure is justified by the following heuristic arguments:

- The optimum prefilter produces a white innovation sequence; therefore, a prefilter which is "near optimum" should have a "nearly white" innovations sequence.

- Replacing the correlated innovations sequence with a white sequence can only increase the required data transmission rate. Therefore the heuristic design technique described above provides an upper bound on the transmission rate required to achieve a specified distortion with a given prefilter.

- If DPCM encoding is used, the prefilter will be effectively modeled as though its innovations sequence were white. Therefore, this heuristic design procedure is consistent with DPCM.

When the output of the prefilter is to be transmitted over a digital channel, the natural choice for an

encoder is a differential pulse code modulator (DPCM) which uses the reduced-order system dynamics model of the prefilter. Therefore, the use of a reduced-order prefilter results in a simplification of both the prefilter and the encoder.

2.4 EXAMPLE OF PREPROCESSOR DESIGN

In this section a simple example is used to illustrate the techniques which can be employed in designing a preprocessor. The example problem shown in Fig. 2.4-1 was chosen to emphasize the reduction in channel capacity which can be achieved by using a data preprocessor. The rms acceleration was taken to be

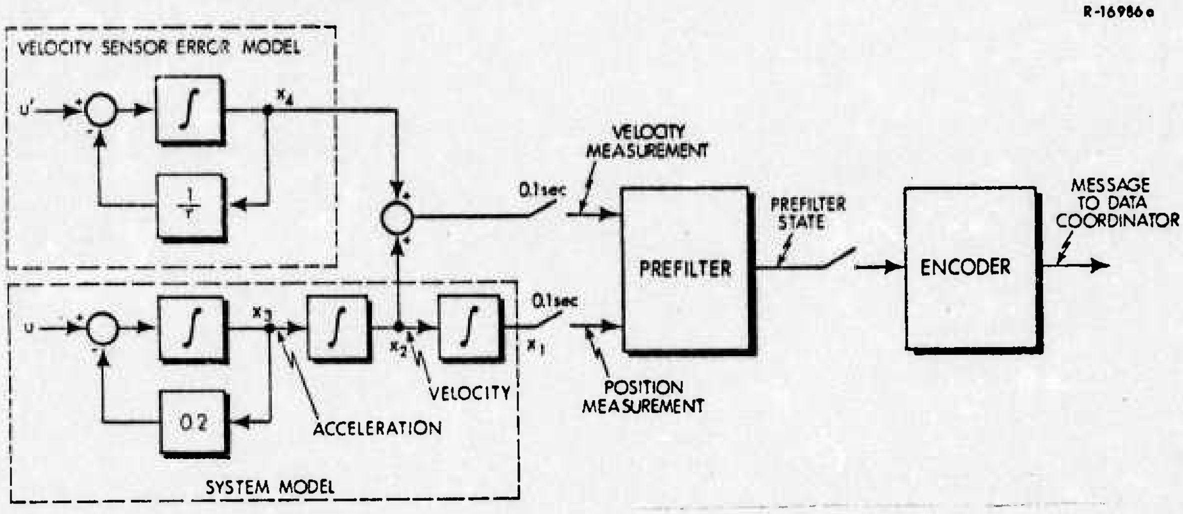

Figure 2.4-1 Preprocessor Design Example

$$\sigma_3 = 120 \text{ ft/sec}^2 \tag{2.4-1}$$

with a correlation time of 5 seconds. The rms initial position and velocity uncertainties were assumed to be

$$\sigma_1(0) = 500 \text{ ft} \tag{2.4-2}$$

and

$$\sigma_2(0) = 50 \text{ ft/sec} \tag{2.4-3}$$

The rms position measurement error was 100 ft and the rms velocity measurement error was 100 ft/sec with a correlation time, τ, of 10 seconds. This example can be considered to be a highly simplified model of a dead-reckoning navigation system for a highly-maneuverable vehicle. The resulting preprocessor may be similar to the type of preprocessor which would be used with a Global Positioning System (GPS) receiver. Since this system is unstable, the number of bits required to transmit whole values of position and velocity grows with time; therefore, some type of preprocessor (e.g. a simple differencing operation) is mandatory. The preprocessor removes redundancy in the data and permits the number of bits required at each transmission time to be bounded. The Kalman filter is the optimum prefilter and is studied first. Suboptimal preprocessors using MVRO prefilters are considered next.

Using a Kalman prefilter to remove redundancy in the measurements and using a scheme such as a Differential Pulse Code Modulation (DPCM) to transmit only the changes in the Kalman filter state greatly reduces the required channel capacity. The techniques discussed in Appendix A were used to compute the distortion introduced by constraints on the channel capacity. Figure 2.4-2 shows the results. It was assumed that the steady-state rms position error was the only measure of distortion. The Kalman prefilter has a steady-state rms

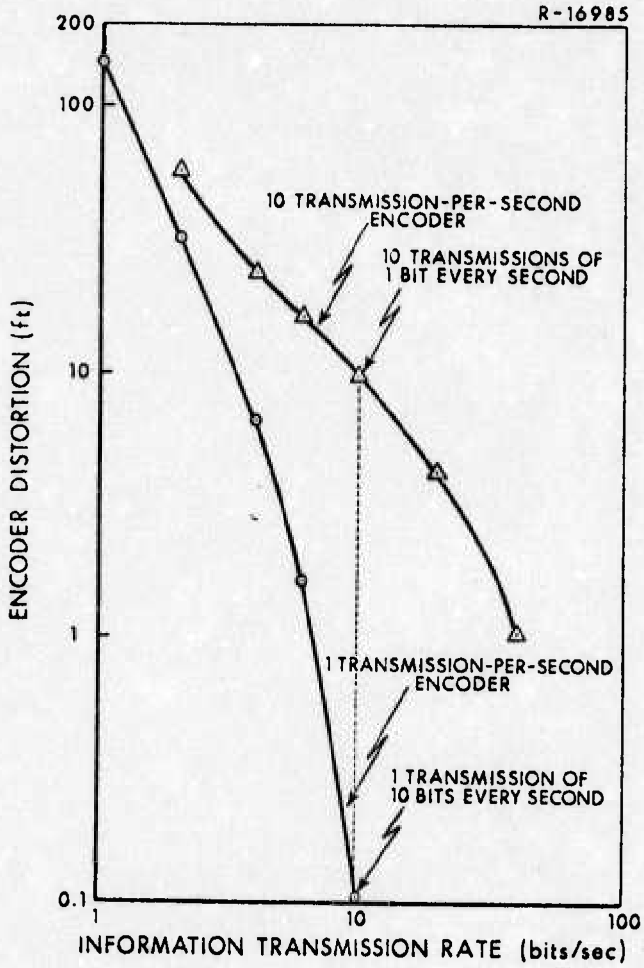

Figure 2.4-2 Additional Preprocessor Distortion vs Transmission Rate

position error of 37.6 ft. The additional distortion introduced by the finite capacity of the preprocessor-to-data-coordinator communications channel is plotted in Fig. 2.4-2. The total rms position error is obtained by root-sum-squaring the number obtained from this graph with the prefilter rms error. Two types of optimum encoders were studied: one which transfers a fixed number of bits every 0.1 second and one which transfers a fixed number of bits every second. The information transmission rate (in bits/second) was varied and the resulting distortion plotted.

Note that in all cases, the one-transmission-per-second encoder introduces less distortion than the ten-transmissions-per-second encoder when the same bit rates are used. This verifies that the prefilter is indeed performing data compression.

Figure 2.4-2 shows that a channel capacity of four bits per second used in conjunction with the one-transmission-per-second encoder adds a negligible distortion to the prefilter distortion. Therefore, this will be taken as the design point. Now a practical encoder which approaches these optimum characteristics will be developed for a digital channel.

In Appendix A, it is shown that the optimum encoder-channel-decoder combination can be modeled as a Kalman filter operating on a set of measurements

$$\underline{z}_n^* = H_n^* \hat{\underline{x}}_n + \underline{v}_n^* \tag{2.4-4}$$

where $\underline{v}_n^*$ is white Gaussian noise with variance R^* and H_n^* is computed by an algorithm described in Section A.3. At the design point chosen for this example, the steady state value of H^* is

$$H^* = \sqrt{R^*}\,[0.1515\ \mathrm{ft}^{-1}, \quad 0, \quad 0, \quad 0] \tag{2.4-5}$$

A DPCM system is chosen for the encoder. Since four bits per second sent at one-second intervals was chosen as the design point, four bits or 16 levels are used for the quantizer. Results from Ref. 14 are summarized in Fig. 2.4-3. This curve gives the ratio of the per-transmission rms quantization error $\sqrt{R^*}$ to the quantization level ℓ as a function of the number of bits used in each transmission. It is assumed that equally spaced quantization steps of optimum length are used and that round-off is performed rather than truncation. For four bits of quantization, this curve shows that

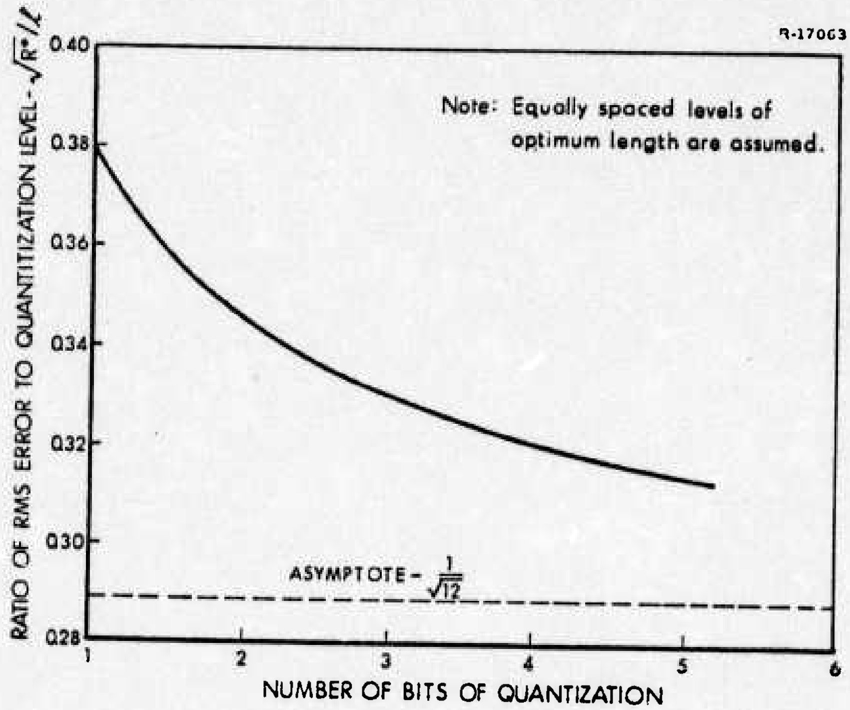

Figure 2.4-3 Ratio of rms Quantization Error to Quantization Level

$$\sqrt{R^*} = 0.320\ell \tag{2.4-6}$$

Therefore,

$$H^* = \ell\,[0.0485\,\text{ft}^{-1}, \quad 0, \quad 0, \quad 0] \tag{2.4-7}$$

The optimum gain for signal reconstruction is

$$K^* = P^* H^{*T}\left[H^* P^* H^{*T} + R^*\right]^{-1} \tag{2.4-8}$$

In this particular case

$$K^* = \begin{bmatrix} 10.3/\ell \\ 10.9/\ell \\ 5.25/\ell \\ 0.00367/\ell \end{bmatrix} \tag{2.4-9}$$

The resulting channel-encoder and channel-decoder are shown in Figs. 2.4-4 and 2.4-5. Note that the dynamics models used do not include the dynamics of the correlated velocity-measurement noise because this state does not need to be estimated by the channel-encoder and channel-decoder.

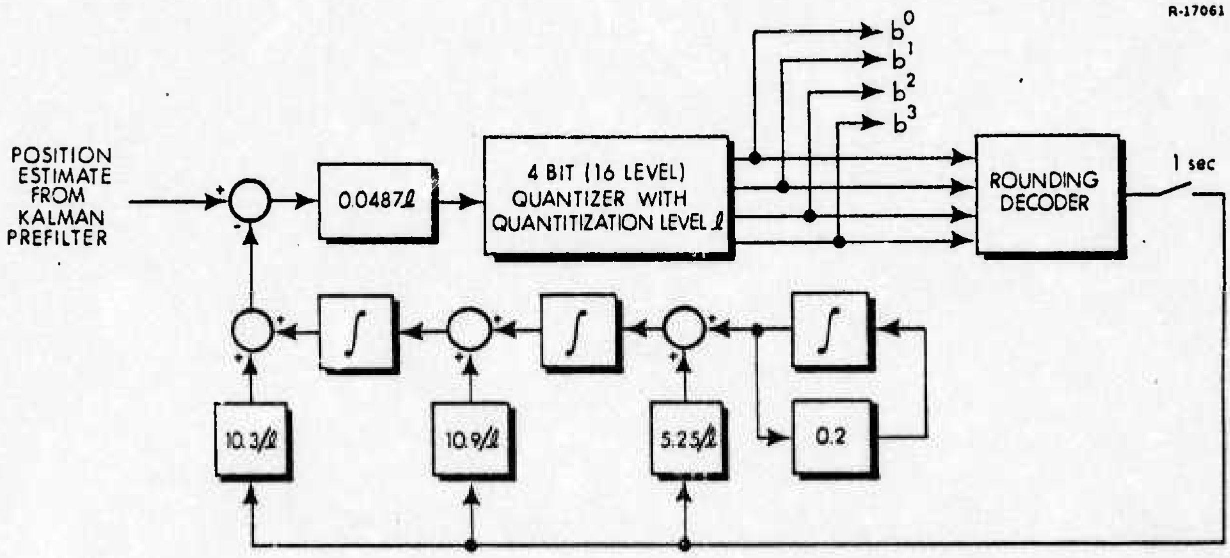

Figure 2.4-4 Example Channel-Encoder

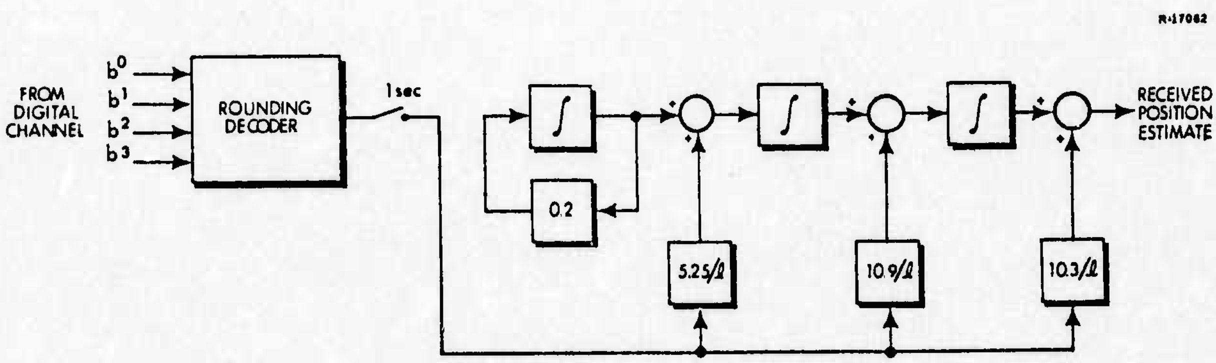

Figure 2.4-5 Example Channel-Decoder

Suboptimal prefilters were also investigated. Figure 2.4-6 compares the rate-distortion functions for the optimal prefilter and two suboptimal prefilters, using optimal decoders. One prefilter models all states except measurement correlation; the other models all states except acceleration correlation. It can be seen that the latter performs very near the optimum and so would be the better choice for a suboptimum prefilter. Of course, achieving the performance shown in Fig. 2.4-6 requires the use of an optimum decoder which models both the system states and the prefilter states. However, the MVRO estimator techniques of Ref. 7 can be used to design simpler decoders which may achieve performance near that shown in Fig. 2.4-6.

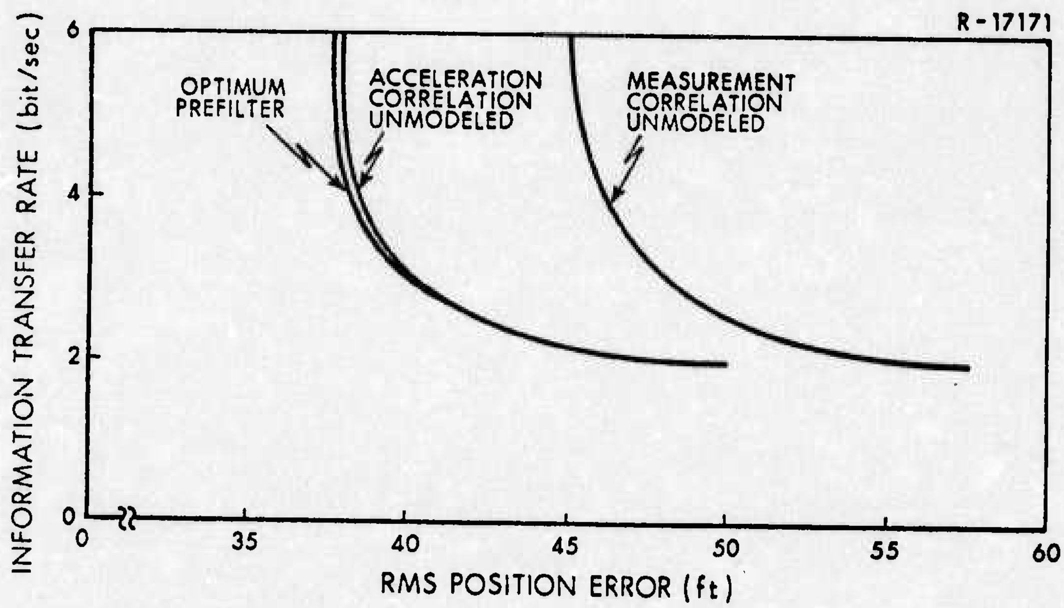

Figure 2.4-6 Comparison of Optimal and Suboptimal Prefilters

3. DATA COORDINATOR DESIGN

3.1 OPTIMAL DATA COORDINATOR DESIGN

Once the preprocessor structure has been fixed, a data coordinator must be designed to combine all of the compressed data and produce estimates of the system states for control and display. Assuming a linear system driven by Gaussian noise adequately models the system, then a Kalman filter preserves all relevant information about the state. Therefore the best data coordinator is a Kalman filter. But it is the Kalman filter for the augmented system which includes the data prefilter dynamics.

If a single sensor with an optimum prefilter has been used, then the data coordinator design becomes particularly simple. Since the innovations sequence in the optimum prefilter (refer to Fig. 2.2-1) is a white Gaussian sequence, the model for the augmented system which produces the preprocessed data sequence can be simplified to the form shown in Fig. 3.1-1. Furthermore, if differential pulse code modulation (DPCM) is used to encode each data channel, then the optimum data coordinator takes the form shown in Fig. 3.1-2.

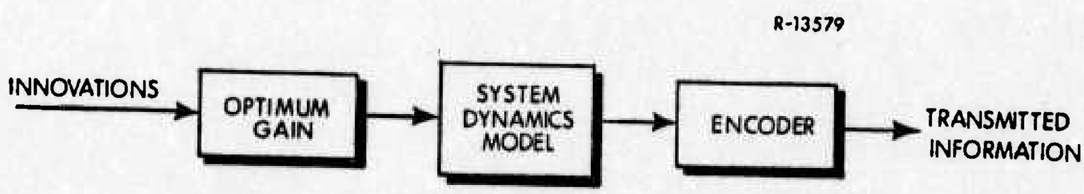

Figure 3.1-1 Simplified Model of Augmented System with Optimum Preprocessor

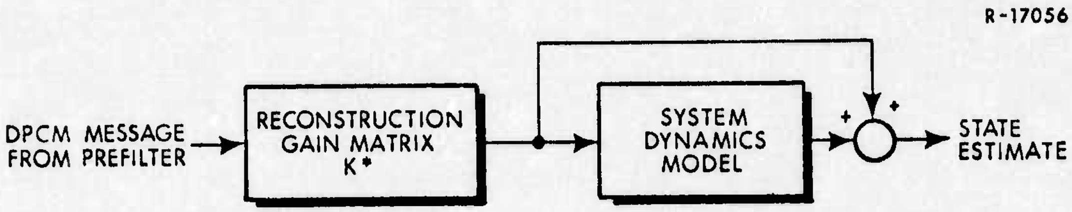

Figure 3.1-2 Optimum Data Coordinator for Optimum Prefiltering and DPCM

If non-optimal prefiltering is used, or if more than one sensor is in operation, then the optimum data coordinator takes the form shown in Fig. 3.1-3. In cases such as this, implementing the optimum data coordinator may require considerable computer resources and would therefore defeat one of the purposes of data preprocessing. Thus a suboptimal data coordinator should be used. In the next section, techniques for

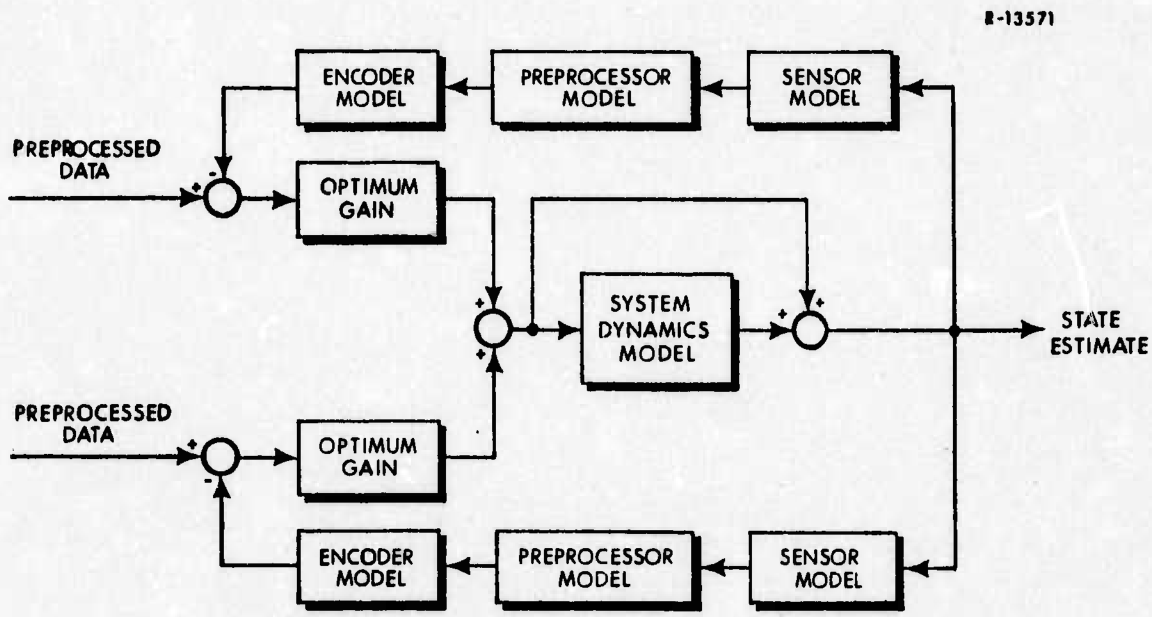

Figure 3.1-3 General Optimum Data Coordinator Structure

designing suboptimal data coordinators which do not exceed computational constraints are discussed.

A heuristic approach to data coordinator design is to consider the data coming from a preprocessor as a measurement of the state

$$\underline{z}^* = H^* \underline{x} + \underline{v}^*$$

where H^* is the encoder compression matrix described in Appendix A and $\underline{v}^*$ is assumed to be a white Gaussian noise composed of prefilter estimation errors and data transmission noise. The trouble with this approach is that the measurement noises are in general <u>not</u> white and may be correlated between channels.

3.2 SUBOPTIMAL DATA COORDINATOR DESIGN

Usually constraints must be imposed on the complexity of the data coordinator so that timing and computer capacity restrictions can be met. The optimum data coordinator requires a dynamics model which includes system, sensor, and preprocessor models and therefore may be too complex to implement. The MVRO estimator design techniques of Ref. 7 can be used to develop a data coordinator of constrained complexity. A MVRO estimator uses, as a memory, a dynamics model for only a portion of the states. Thus, any constraints on complexity can be met by a MVRO estimator with a suitably chosen memory. For a specific selection of memory states, the MVRO estimator produces the most accurate estimate of the states required for display and control. In some cases, it may be possible to use a MVRO estimator as the data coordinator; in other cases, it may not be practical to implement the MVRO estimator due to the complexity of the computations required to compute the optimum estimator parameters.

In any case, the MVRO estimator design techniques provide guidelines for designing and evaluating data coordinators.

3.3 DATA COORDINATOR EVALUATION

Like a data preprocessor, a data coordinator can be evaluated by determining how much information about the state has been sacrificed to meet complexity constraints. This sacrificed information is reflected by an increase in the mean-square estimation error. Since it can usually be assumed that linear dynamics models apply, computer programs which have been developed to evaluate reduced-order estimators can be used to determine the least squares estimation error of a particular data coordinator-preprocessor combination. Note that the data coordinator cannot be evaluated independently of the data preprocessors. The performance of the data coordinator depends on the particular preprocessors used. The next section presents techniques which can be used to evaluate a suboptimal data coordinator.

Although the software used in the data coordinator depends on the particular sensors and preprocessors used, the hardware design of the data coordinator-preprocessor interface can be standardized. This is because the bit-rates and transmission intervals depend mainly on the general type of sensor (e.g. a velocity sensor) and on the system dynamics. Therefore, electrical and mechanical interfaces can be standardized, and only software changes need be made when a sensor is changed.

4. SENSITIVITY ANALYSIS OF REDUCED-ORDER ESTIMATORS

4.1 ANALYSIS PROCEDURE

A reduced-order estimator is a data processor which uses a memory of lower order than that required for optimal estimation. In a modular estimator design, reduced-order estimators may be used for prefilters and will almost certainly be used for the data coordinator. The performance of these reduced-order estimators depends on the signals they receive as inputs. In this section, the sensitivity of reduced-order estimators to changes in their input signal structure is investigated. The investigation consists of the following steps:

- A reduced-order estimator is designed based on a fixed model, called the design model.
- A different model, called the reference model, is used to describe a possible real world situation.
- The covariance of the error in the state estimate is computed using the equations derived in this report.
- The performance of the reduced-order estimator for a class of reference models is compared with that of other estimators and with performance specifications.

The techniques presented in this section can be used to analyze prefilter designs to determine their sensitivity to changes in the system model. However, the most important use of these techniques is in analyzing the performance of the data coordinator and thereby the performance of the

complete modular estimator. To analyze the data coordinator, the reference system is the augmented system which includes the prefilters. The measurements are the outputs of the encoder-channel-decoder combination and so take the form of linear combinations of the augmented states corrupted by white Gaussian noise. The prefilter designs are assumed fixed and the system model is varied to determine the sensitivity of the data coordinator. If the sensitivity is too great, not only the data coordinator but also the prefilters may need to be redesigned.

4.2 ERROR COVARIANCE EQUATIONS

Both the design model and the reference model take the form shown in Fig. 4.2-1. The system state is assumed to satisfy the linear vector difference equation

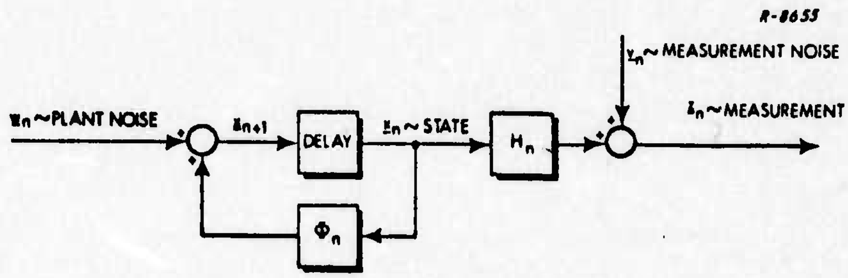

Figure 4.2-1 Form of Design and Reference Models

$$\underline{x}_{n+1} = \Phi_n \underline{x}_n + \underline{w}_n \qquad (4.2\text{-}1)$$

where

$\underline{x}_n$ is the state at time n

$\underline{w}_n$ is a vector of zero mean white noises

and Φ_n is the state transition matrix.

The measurements $\underline{z}_n$ are assumed to be related to the state of the system by the linear vector equation

$$\underline{z}_n = H_n \underline{x}_n + \underline{v}_n \tag{4.2-2}$$

where $\underline{v}_n$ is a vector of zero mean white noises.
and H_n is the measurement matrix.

The state estimate is assumed to take the form

$$\hat{x}_{n|n} = C_n \underline{m}_n + K_n \underline{\nu}_n \tag{4.2-3}$$

where

$\underline{m}_n$ is a memory vector (usually of reduced order)

$\underline{\nu}_n$ is the new information (innovations) supplied by the n^{th} measurement $\underline{z}_n$

Here $\underline{m}_n$ is assumed to satisfy the difference equation

$$\underline{m}_{n+1} = A_n \underline{m}_n + B_n \underline{\nu}_n \tag{4.2-4}$$

and $\underline{\nu}_n$ is given by

$$\underline{\nu}_n = \underline{z}_n - G_n \underline{m}_n \tag{4.2-5}$$

The matrices A_n, B_n, C_n, G_n, and K_n completely define the reduced-order estimator.

The prime concern of estimator performance analysis is to determine P_n, the covariance matrix for the filtering error

$$\tilde{\underline{x}}_{n|n} = \underline{x}_n - \hat{\underline{x}}_{n|n} \tag{4.2-6}$$

However, to save computations, the error covariance matrix for the one-step prediction error is computed first and then used to compute the filtering error covariance matrix. The one-step prediction error is defined to be

$$\tilde{\underline{x}}_{n+1|n} = \underline{x}_{n+1} - C_{n+1} \underline{m}_{n+1} \tag{4.2-7}$$

In Appendix B it is shown that

$$P_n = \Psi_n \Pi_n \Psi_n + K_n R_n K_n^T \qquad (4.2\text{-}8)$$

where

$$\Psi_n = \left[(I - K_n H_n) \; , \quad K_n (G_n - H_n C_n) \right] \qquad (4.2\text{-}9)$$

and

$$\Pi_n = E \begin{bmatrix} \underline{\tilde{x}}_{n|n-1} \underline{\tilde{x}}^T_{n|n-1} & \underline{\tilde{x}}_{n|n-1} \underline{m}^T_n \\ \underline{m}_n \underline{\tilde{x}}^T_{n|n-1} & \underline{m}_n \underline{m}^T_n \end{bmatrix} \qquad (4.2\text{-}10)$$

Furthermore, Appendix B shows that Π_n satisfies the difference equation

$$\Pi_{n+1} = \Delta_n \Pi_n \Delta_n^T + \Lambda_n R_n \Lambda_n^T + \begin{bmatrix} Q_n & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.2\text{-}11)$$

where the matrices Δ_n and Π_n are defined in Appendix B.

Therefore, the computation of the error covariance involves two steps.

(1) The recursion equation for Π_n is mechanized. A formula for the starting value Π_0 is given in Appendix B.

(2) The estimation error covariance matrix P_n is computed as needed.

In the next section, some simple examples are used to illustrate the types of sensitivity analysis studies which may be performed using these error covariance equations.

4.3 EXAMPLES

The tracking problem illustrated in Fig. 4.3-1 was chosen as the basis for all the examples in this section. This tracking problem can be viewed as a highly simplified model of a dead-reckoning navigation system for a highly-maneuverable vehicle. The acceleration x_3 is assumed to be a correlated Gaussian noise process. The velocity x_2 results from integrating the acceleration, and the position x_1 is simply the integral of velocity.

The rms value of the acceleration process was taken to be

$$\sigma_3 = 120 \text{ ft/sec}^2 \tag{4.3-1}$$

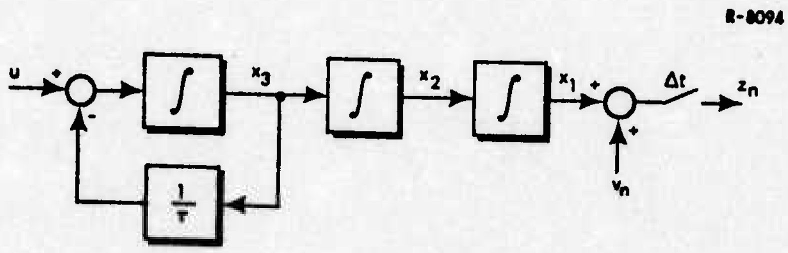

Figure 4.3-1 Tracking Problem

The rms value of the initial position uncertainty was assumed to be

$$\sigma_1(0) = 500 \text{ ft} \tag{4.3-2}$$

and the rms value of the initial velocity uncertainty was chosen to be

$$\sigma_2(0) = 50 \text{ ft/sec} \tag{4.3-3}$$

Different examples were created by assuming different measurement mechanisms. In all examples, measurements were assumed to be taken at 0.2 second intervals.

As a first example, it was assumed that a measurement of position was available. This measurement was corrupted by an additive, white, Gaussian noise v_n with 100 ft standard deviation. The sensitivity of estimator performance to variations in the acceleration correlation time τ' were investigated. A two-state estimator which used a memory of position and velocity only was designed using the MVRO observer-estimator techniques with the design value of τ' equal to five seconds. The sensitivity of this estimator was compared to that of a three-state estimator designed for τ' equal to five seconds. Figure 4.3-2 shows a comparison of the rms position error at 25 seconds.

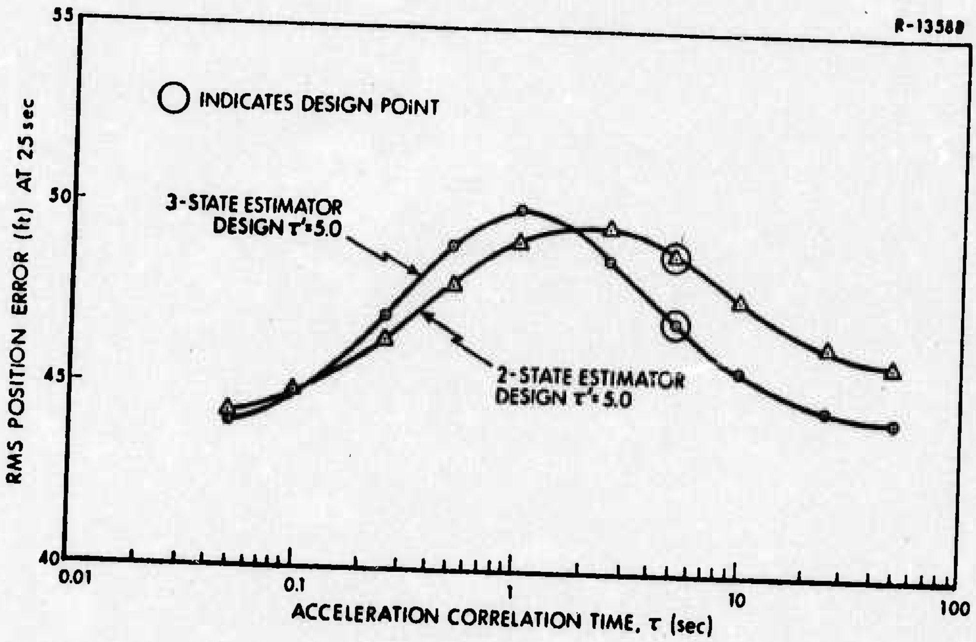

Figure 4.3-2 Sensitivity of Estimator to Changes in Acceleration Correlation Time

Since the three-state estimator is the Kalman filter for the design model, the three state estimator performs better than the two-state estimator at the design point. However, as the acceleration correlation time in the reference

model is decreased, the two curves cross. For acceleration correlation times between 0.1 and 1.0 seconds, the two-state estimator performs better than the three-state estimator. Furthermore, the peak of the two-state estimator curve is lower than the peak of the three-state estimator curve. Therefore, considering worst case performance for the acceleration correlation time in the range shown in Fig. 4.3-2, the two-state estimator is preferable to the three-state estimator. Of course, this does not mean that the two-state estimator tested here is preferable to all three-state estimators. It means that, for the design model chosen, the two-state estimator is less sensitive to errors in modeling the acceleration correlation time. When the acceleration correlation time is shorter than expected, the performance of the three-state estimator degenerates because it relies too heavily on its memory of acceleration.

A second set of examples was studied using the same basic system model modified as shown in Fig. 4.3-3. The acceleration correlation was time fixed at 5.0 seconds, and in addition to the position measurement, a velocity measurement was taken every 0.1 seconds. This velocity measurement was assumed to be corrupted by a zero-mean, Gaussian measurement noise with a standard deviation of 100 ft/sec and correlation time τ. An additional state must be added to the system model to account for this correlated measurement noise. Therefore, a four-state estimator is required to implement the Kalman filter for a given design model of this system. Two design points were chosen: $\tau = 0.1$ seconds and $\tau = 10$ seconds. The performance of the resulting four-state estimators was compared to that of two corresponding three-state estimators designed using MVRO techniques.

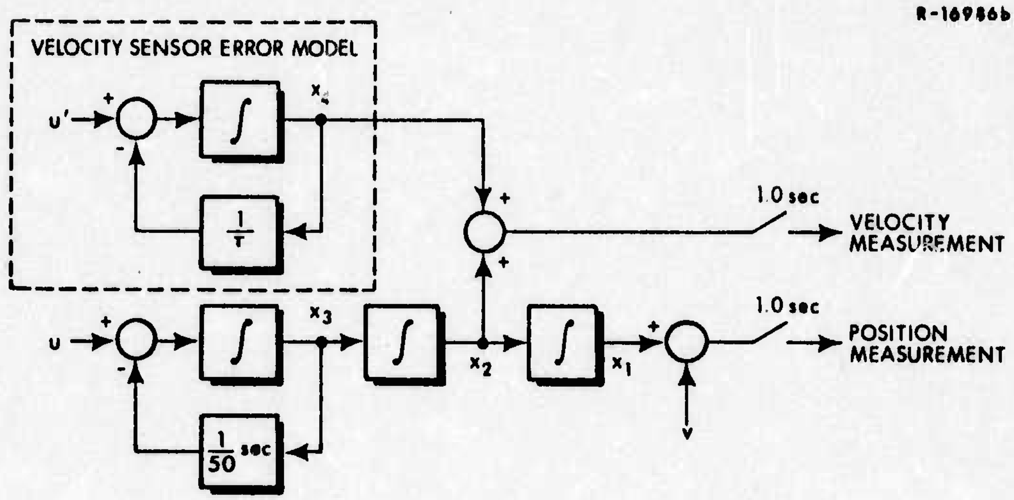

Figure 4.3-3 Tracking Problem with Velocity and Position Measurements

The three-state estimators used a memory of position, velocity and acceleration, but did not attempt to "remember" the velocity measurement error. Figure 4.3-4 shows the variation in steady state rms position error for these estimators as the correlation time of the velocity measurement error is changed in the reference model. Figure 4.3-5 shows the variation of rms velocity error. If it is believed that the velocity-measurement-error correlation time may vary over the indicated range of values, then the estimator whose performance curve has the lowest peak is most desirable. Adopting this criterion, the four-state estimator with design $\tau = 10.0$ seconds has the best position error performance. However, the three-state estimator with design $\tau = 10.0$ performs very nearly as well in position error. Furthermore, the latter estimator performs much better in terms of velocity error. Therefore, the three-state estimator with design $\tau = 10.0$ seconds would probably be the most desirable of the four estimators tested.

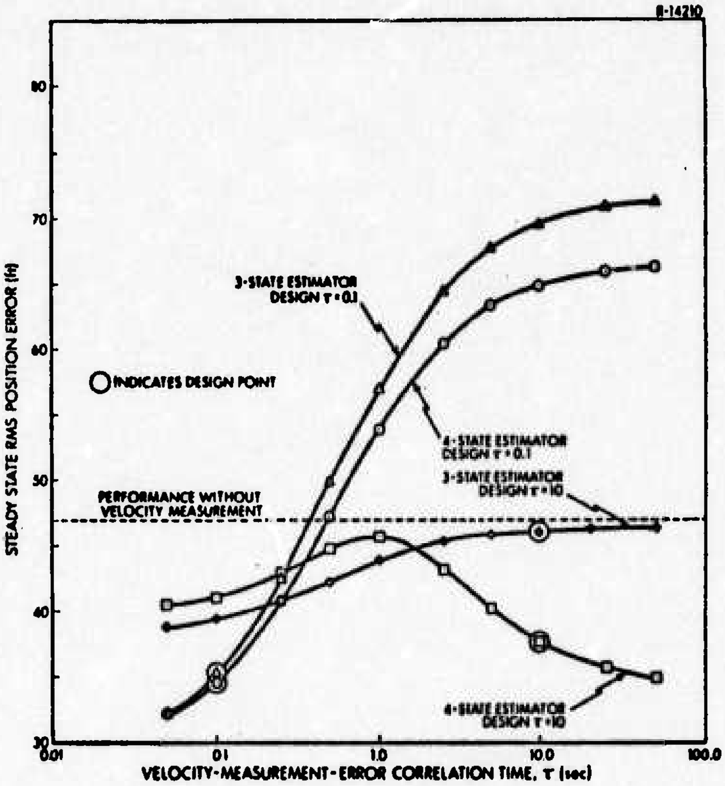

Figure 4.3-4 Sensitivity of Position Error

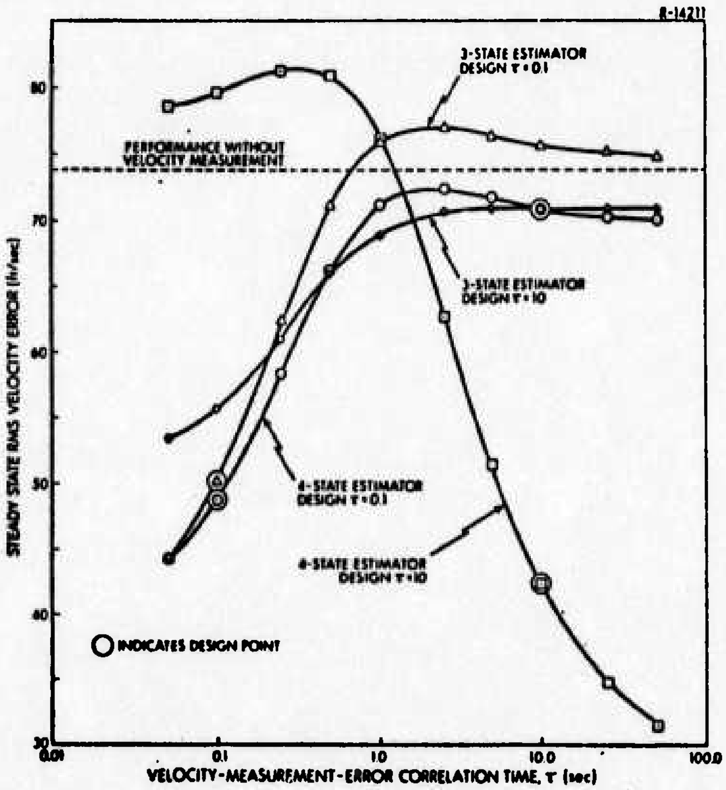

Figure 4.3-5 Sensitivity of Velocity Error

Other conclusions can be drawn from these curves. For example, consider the curves for the four-state estimator with design $\tau = 10.0$ seconds. Note that a mismodeling of the velocity-measurement-error correlation times does not greatly affect the position error performance and that in all cases the position error performance is superior to the performance without the velocity measurement. However, it can be seen that mismodeling does greatly affect the velocity error performance. In fact, for correlation times much shorter than the design correlation time, the velocity error performance is worse than if no velocity measurement were taken.

Looking now at the performance of the four-state estimator with design $\tau = 0.1$ seconds, the affect of mismodeling on velocity error is rather large, but in all cases taking the velocity measurement does improve performance. However, the position error performance for this estimator can be significantly degraded by taking the velocity measurement if the correlation time is mismodeled.

For the three-state estimator with design $\tau = 0.1$ seconds, both position and velocity error performance are degraded by taking velocity measurements and mismodeling the measurement-error-correlation time. Of the four estimators considered here, only the three-state estimator with design $\tau = 10.0$ seconds uses velocity measurements to improve both position and velocity error performance even when the measurement error correlation time is badly mismodeled.

It should be noted that the minimax design procedures of Refs. 15 and 16 can be used in some cases to determine the estimator which has the least sensitivity to modeling errors of any estimator using the same order memory as the system. Unfortunately, these procedures cannot be feasibly implemented

for these examples, and have not yet been extended to reduced-order estimators. The last example seems to indicate that some three-state estimator may be less sensitive to modeling errors than any four-state estimator. However, the results presented here are certainly not conclusive.

5. SUMMARY

In the past, navigation data processing has typically been performed in a central data processor, as illustrated in Fig. 5-1. The development of microprocessors and other Large Scale Integration (LSI) devices has made a new data processing structure feasible. Called modular estimation in this report, this data processing structure is composed of data preprocessors located at the sensors and a data coordinator located with the user, as shown in Fig. 5-2.

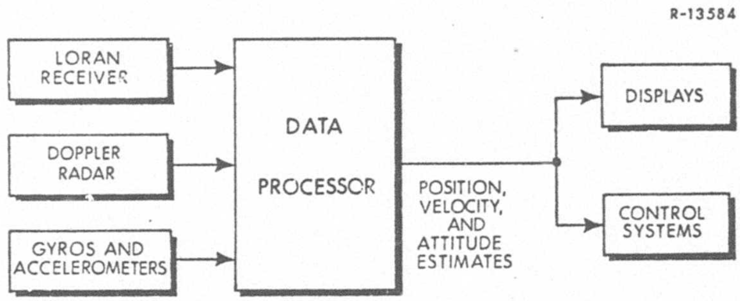

Figure 5-1 Centralized Navigation Data Processor

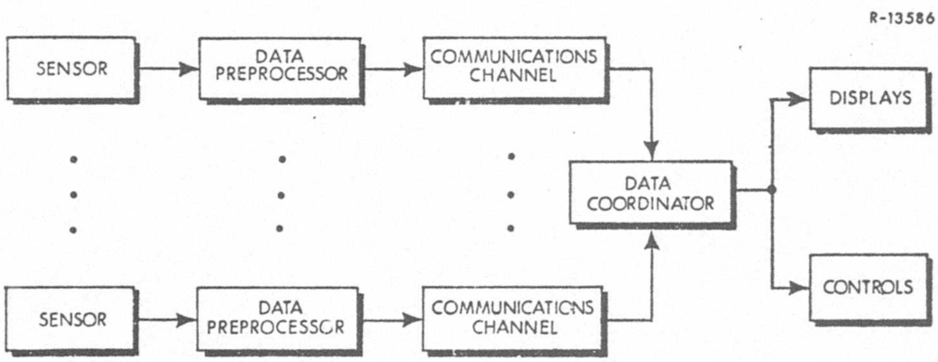

Figure 5-2 Modular Navigation Data Processor

Current trends toward integrating all avionics in a coordinated aircraft information system make modular estimators desirable for the following reasons:

(1) Computer Capacity — Data preprocessors operate in parallel with the data coordinator. This increases the computational capacity of the system without requiring an increase in the speed of the data coordinator; and without introducing some of the timing problems usually associated with real-time data processing.

(2) Executive Program Complexity — Since data can be stored in the preprocessors, many of the complex timing problems and executive interrupt structures usually associated with real-time data processing can be avoided.

(3) Channel Capacity - Since data compression occurs at the sensor, the capacity required for the communication channel feeding the data coordinator can be greatly reduced.

(4) Modularity and Flexibility — The interfaces between the data coordinator and certain functional types of sensor/preprocessor packages can be fixed. Thus, a velocity reference could replace any other velocity reference without affecting the hardware design of any other part of the system.

(5) System Reliability — Preprocessors can be designed in such a way that a failure of a single component will still leave the system operational.

This report presents methods of designing and evaluating modular estimators. An example is used to illustrate the techniques which can be employed in designing a preprocessor. The sensitivity analysis techniques required to evaluate the performance of the complete modular estimator are also illustrated.

This design procedure can solve many of the problems encountered in developing a modular estimator. For example, with a data bus system like the Digital Avionics Information System (DAIS), this design procedure can be used to answer questions such as the following:

- How many bits per second should be allocated for each sensor?
- How often should sensor data be put on the bus?
- What type of prefilter should be used with each sensor?
- How should sensor data be quantized and coded for transmission?
- What type of algorithm should be used to coordinate the data coming from the sensors?

APPENDIX A

RELEVANT INFORMATION THEORY CONCEPTS

A.1 MODULAR ESTIMATION AND COMMUNICATION SYSTEM DESIGN

In some respects, modular estimator design can be viewed as a communication system design problem; therefore, several concepts which are used extensively in communication system design can be applied to the analysis and design of modular estimation algorithms. This section explores the connections between modular estimation and communication system design. Subsequent sections present important information theory concepts and discuss their application to modular estimator design.

The classical communication system configuration is shown in Fig. A.1-1. Messages are produced by the source and are to be transmitted over the channel to the user. An encoder is used to translate the messages to a form which can be transmitted over the channel; a decoder performs the inverse translation back to a form which the user can interpret. During message transmission, the channel introduces errors; so the received message $\hat{\underline{x}}$ differs from the transmitted message $\underline{x}$. A distortion measure $p(\underline{x},\hat{\underline{x}})$ is used to

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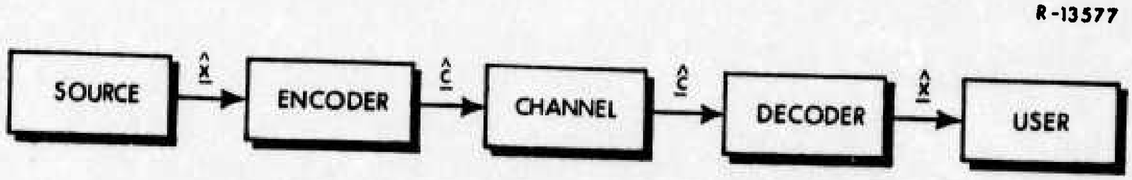

Figure A.1-1 Classical Communication System Configuration

measure the significance of this deviation. By properly designing the encoder and decoder to match the source and channel, the expected distortion $E\left[p(\underline{x},\hat{\underline{x}})\right]$ can be minimized.

Usually the communication system design can be partitioned as illustrated in Fig. A.1-2. By separating the encoder into a source encoder and a channel encoder and the decoder into a channel decoder and a source decoder, the design procedure can be factored into a phase which depends primarily on the channel characteristics and a phase which depends primarily on the source characteristics. The design of the source encoder-decoder combination will be emphasized here; channel encoder-decoder design will not be discussed in detail. The only relevant characteristic of the channel will be its capacity which will be defined later.

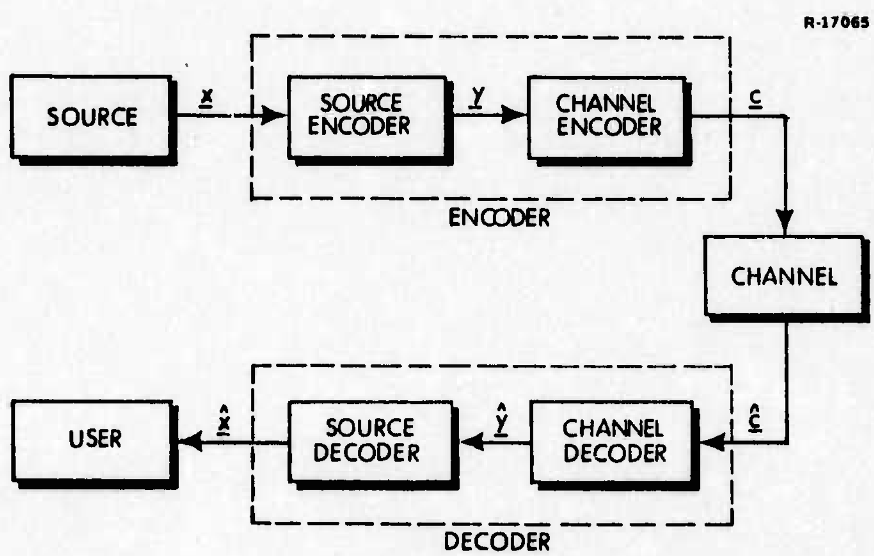

Figure A.1-2 Partitioned Communication System Design

The modular estimation problem can be cast in the communication system configuration shown in Fig. A.1-3. In the estimation problem, the encoder is composed of the sensor and the preprocessor. Therefore part of the encoder

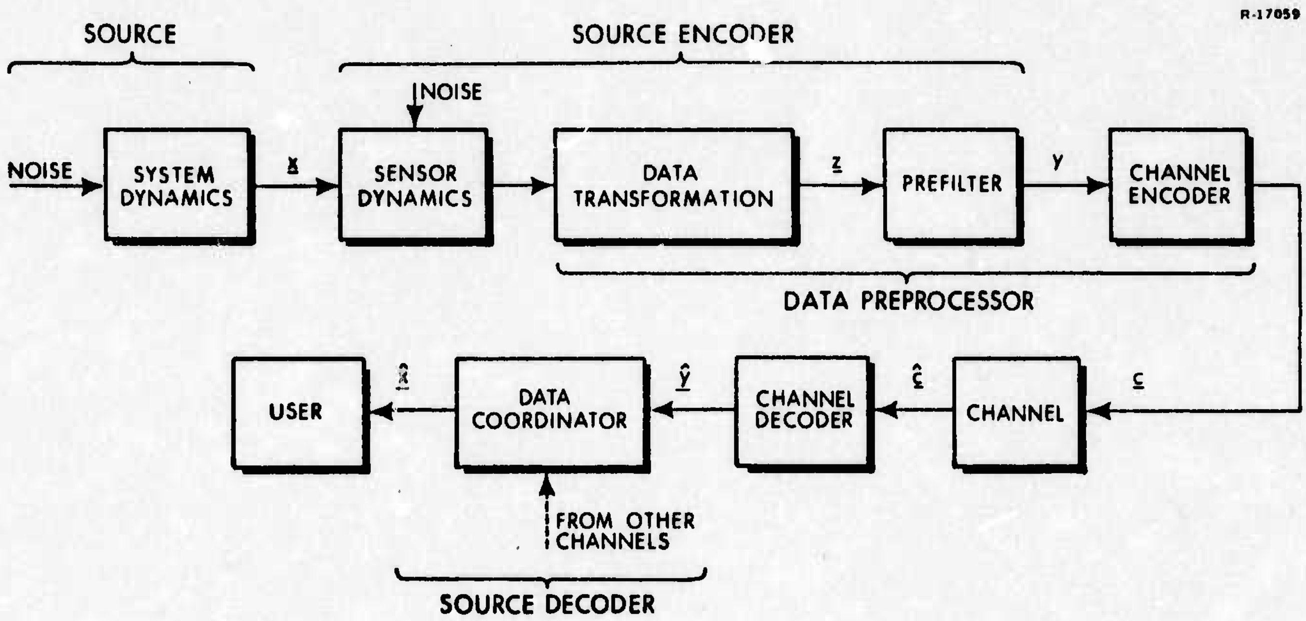

Figure A.1-3 Communications System Interpretation of Modular Estimator

design, the sensor, is usually fixed and not subject to optimization. This requires some minor modification of standard information theory methods. The details of this development will follow. First relevant terms from information theory will be defined. More complete discussions of these concepts are contained in Refs. 8 through 10.

A.2 DEFINITION OF TERMS

A.2.1 Entropy

The entropy of a random variable is a measure of the uncertainty in the value which it may assume. For a discrete random variable x which may assume k values $a_1, a_2, \ldots, a_k$ with the probabilities $p(a_1), p(a_2), \ldots, p(a_k)$ the entropy is defined to be

$$H(x) = E[-\log p(x)]$$

$$= -\sum_{i=1}^{k} p(a_i) \log p(a_i) \qquad \text{(A.2-1)}$$

If y is another variable taking n values $b_1, b_2, \ldots, b_n$ which is jointly distributed with x, then the entropy of the joint distribution is

$$H(x,y) = E[-\log p(x,y)]$$

$$= -\sum_{i=1}^{k} \sum_{j=1}^{n} p(a_i, b_j) \log p(a_i, b_j) \qquad \text{(A.2-2)}$$

where $p(a_i, b_j)$ is the probability that x takes the value a_i and y takes the value b_j. The <u>conditional entropy</u> of x given y is defined to be the expected value of the entropy of the conditional distribution for x given y. That is

$$H(x|y) = -\sum_{j=1}^{n} p(b_j) \sum_{i=1}^{k} p(a_i|b_j) \log p(a_i|b_j)$$

$$= E[-\log p(x|y)] \qquad \text{(A.2-3)}$$

where $p(a_i|b_j)$ is the probability that x takes the value a_i given that y has the value b_j. The conditional entropy measures the average uncertainty about the value of x which remains after the value assumed by y is known.

The entropy function has a number of properties which justify its interpretation as a measure of uncertainty.

(1) H(x) is a function of the probabilities $p(a_1)$, $p(a_2)$, ..., $p(a_k)$ only and not a function of the values which x may assume. The uncertainty depends

only on the probabilities of the alternatives and not on any other characteristic of the alternatives.

(2) $H(x)$ is a continuous function of the $p(a_i)$'s. A small change in the probabilities produces only a small change in the uncertainty.

(3) For fixed k, $H(x)$ achieves its maximum when

$$p(a_i) = \frac{1}{k} \qquad i = 1,2,\dots, k \qquad \text{(A.2-4)}$$

Uncertainty is maximum when all possible alternatives are equally likely.

(4) Adding an additional value a_{k+1} which x could assume with probability zero does not change the entropy. The addition of zero probability alternatives does not change the uncertainty.

(5) $$H(x,y) = H(y) + H(x|y) \qquad \text{(A.2-5)}$$

The uncertainty in x and y considered jointly is equal to the uncertainty in y alone plus the uncertainty in x given y.

In Ref. 11, Khinchin shows that these properties can be taken as axioms and the form of the entropy function will be deduced to be

$$H(x) = -\lambda \sum_{i=1}^{k} p(a_i) \log p(a_i) \qquad \text{(A.2-6)}$$

where λ is an arbitrary multiplying constant. Therefore these properties uniquely specify the form of the entropy function to within a constant multiplying factor. Changing the multiplying factor effectively changes the base of the logarithm used. Therefore, any choice of logarithmic base is valid as long as the same base is used consistently.

The entropy concept can be extended to continuous random vectors by the techniques explained in Ref. 12. The

entropy of a random vector $\underline{x}$ with probability density function $p(\underline{x})$ is defined to be

$$H(x) = E[-\log p(\underline{x})]$$

$$= -\int_{-\infty}^{\infty} p(\underline{x}) \log p(\underline{x})\, d\underline{x} \tag{A.2-7}$$

If $\underline{x}$ and $\underline{y}$ have the joint density function $p(\underline{x},\underline{y})$ and the conditional density function $p(\underline{x}|\underline{y})$, then the joint entropy of $\underline{x}$ and $\underline{y}$ is

$$H(\underline{x},\underline{y}) = E[-\log p(\underline{x},\underline{y})]$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\underline{x},\underline{y}) \log p(\underline{x},\underline{y})\, d\underline{x}\, d\underline{y} \tag{A.2-8}$$

and the <u>conditional entropy</u> of $\underline{x}$ given $\underline{y}$ is

$$H(\underline{x}|\underline{y}) = E[-\log p(\underline{x}|\underline{y})]$$

$$= \int_{-\infty}^{\infty} \left[-\int_{-\infty}^{\infty} p(\underline{x}|\underline{y}) \log p(\underline{x}|\underline{y})\, dx\right] p(\underline{y})\, d\underline{y} \tag{A.2-9}$$

All of the properties of the entropy function extend to random vectors with continuous distributions.

The entropy of a Gaussian random vector is of particular interest. Since the density function of a Gaussian vector of dimension n with mean $\underline{\mu}$ and covariance P is

$$p(\underline{x}) = \frac{1}{(2\pi)^{n/2}(\det P)^{1/2}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu})^T P^{-1}(\underline{x}-\underline{\mu})\right] \tag{A.2-10}$$

The entropy of $\underline{x}$ is

$$H(\underline{x}) = E\left[-\log p(\underline{x})\right]$$

$$= \frac{n}{2} \log 2\pi + \frac{1}{2} \log \det P$$

$$+ \frac{1}{2} E\left[(\underline{x}-\underline{m})^T P^{-1} (\underline{x}-\underline{m})\right] \log e \qquad \text{(A.2-11)}$$

Noting that

$$E\left[(\underline{x}-\underline{m})^T P^{-1}(\underline{x}-\underline{m})\right] = \mathrm{tr}\left\{P^{-1} E\left[(\underline{x}-\underline{m})(\underline{x}-\underline{m})^T\right]\right\}$$

$$= n \qquad \text{(A.2-12)}$$

gives

$$H(\underline{x}) = \frac{n}{2} \log 2\pi e + \frac{1}{2} \log \det P \qquad \text{(A.2-13)}$$

A.2.2 Average Mutual Information

The amount of information conveyed about a random variable by observing another random variable is measured by the average mutual information. For random vectors $\underline{x}$ and $\underline{y}$ with continuous distributions, the average mutual information is defined to be

$$I(\underline{x};\underline{y}) = E\left[\ln \frac{p(\underline{x},\underline{y})}{p(\underline{x})\, p(\underline{y})}\right] \qquad \text{(A.2-14)}$$

It can be shown that

$$I(\underline{x};\underline{y}) = H(\underline{x}) - H(\underline{x}|\underline{y})$$

$$= H(\underline{x}) + H(\underline{y}) - H(\underline{x},\underline{y}) \qquad \text{(A.2-15)}$$

$$= H(\underline{y}) - H(\underline{y}|\underline{x})$$

Thus the average mutual information can be interpreted as the uncertainty in $\underline{x}$ minus the average uncertainty in $\underline{x}$ given knowledge of the value assumed by $\underline{y}$, i.e., the uncertainty in $\underline{x}$ resolved by the knowledge of $\underline{y}$. This interpretation also applies if the roles of $\underline{x}$ and $\underline{y}$ are interchanged; therefore, this function is called the average <u>mutual</u> information.

The average mutual information of Gaussian vectors and random processes are of particular interest for estimation problems. Reference 12 proves the following results.

(1) Let $\underline{x}$ and $\underline{y}$ be jointly Gaussian random vectors, with joint covariance matrix

$$C = \begin{bmatrix} A & D \\ D^T & B \end{bmatrix} \tag{A.2-16}$$

Then

$$\begin{aligned} I(\underline{x};\underline{y}) &= -\frac{1}{2} \log \det \left(I - D^T A^{-1} D B^{-1}\right) \\ &= -\frac{1}{2} \log \det \left(I - D B^{-1} D^T A^{-1}\right) \end{aligned} \tag{A.2-17}$$

(2) If $y^k = \{\underline{y}_j,\ 0 \le j \le k\}$ is a Gaussian random process and $\underline{x}$ is a Gaussian random vector, then

$$I(\underline{x};y^k) = \frac{1}{2} \log \det E\left[\underline{x}\ \underline{x}^T\right] - \frac{1}{2} \log \det P_k \tag{A.2-18}$$

where

$$P_k = E\left[(\underline{x}-\hat{\underline{x}}_k)(\underline{x}-\hat{\underline{x}}_k)^T\right]$$

$$\hat{\underline{x}}_k = E\left[\underline{x} \mid y^k\right]$$

= minimum variance estimate of $\underline{x}$ given the values of y^k

Some of the more important properties of the average mutual information are stated below:

(1) $$I(\underline{x};\underline{y}) \geq 0 \qquad \text{(A.2-19)}$$

with equality if and only if $\underline{x}$ and $\underline{y}$ are statistically independent. If $\underline{x}$ and $\underline{y}$ are independent, the occurrence of $\underline{y}$ does not reduce the uncertainty in $\underline{x}$.

(2) $$I(\underline{x};\underline{y}) = I(\underline{y};\underline{x}) \qquad \text{(A.2-20)}$$

The information provided about $\underline{x}$ by $\underline{y}$ is equal to the information provided about $\underline{y}$ by $\underline{x}$.

(3) If $\underline{x} = (\underline{x}_1, \underline{x}_2)$ and $\underline{y} = (\underline{y}_1, \underline{y}_2)$ are statistically independent, then

$$I\big((\underline{x}_1,\underline{y}_1);(\underline{x}_2,\underline{y}_2)\big) = I(\underline{x}_1;\underline{x}_2) + I(\underline{y}_1;\underline{y}_2) \qquad \text{(A.2-21)}$$

Statistically independent problems can be treated separately.

(4) If $\underline{y}$ is a vector and $f(\cdot)$ is a compatible function, then

$$I(\underline{x};\underline{y}) \geq I(\underline{x};f(\underline{y})) \qquad \text{(A.2-22)}$$

A transformation of an observation cannot increase the information.

The most important applications of the average mutual information are in determining channel capacity and the rate distortion function of a source. These concepts are defined in the next sections.

A.2.3 Channel Capacity

In designing a communication system, it is useful to know the ultimate information carrying capacity of a channel

with the optimum encoder-decoder combination. The average mutual information can be used to define such an information capacity measure. Let $\underline{c}$ be the input to the channel and let $\hat{\underline{c}}$ be the output. The source-encoder combination which produces $\underline{c}$ determines the probability density function $p(\underline{c})$, but the channel determines the conditional density $p(\hat{\underline{c}}|\underline{c})$.

The average amount of information carried by the channel is measured by the mutual information between $\underline{c}$ and $\hat{\underline{c}}$ which is

$$I(\underline{c};\hat{\underline{c}}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\underline{c}|\hat{\underline{c}})\ p(\underline{c}) \log \left[\frac{p(\hat{\underline{c}}|\underline{c})}{p(\hat{\underline{c}})}\right] d\underline{c}\ d\hat{\underline{c}} \tag{A.2-23}$$

where

$$p(\hat{\underline{c}}) = \int_{-\infty}^{\infty} p(\hat{\underline{c}}|\underline{c})\ p(\underline{c})\ d\underline{c} \tag{A.2-24}$$

The channel capacity is determined by varying $p(\underline{c})$ to maximize $I(\underline{c};\hat{\underline{c}})$. That is,

$$C = \max_{p(\underline{c})} I(\underline{c};\hat{\underline{c}}) \tag{A.2-25}$$

Thus the channel capacity determines the maximum amount of mutual information which can be achieved if the encoder is designed to present the optimum message structure to the channel.

A.2.4 The Rate-Distortion Curve

For a given source, each encoder-channel-decoder combination will have an information transfer rate which is less than or equal to the channel capacity. Another measure of

the performance of the encoder-channel-decoder combination is distortion $E[\rho(\underline{x},\hat{\underline{x}})]$ where $\underline{x}$ is the message which was transmitted, $\hat{\underline{x}}$ is the estimate received by the user, and $\rho(\underline{x},\hat{\underline{x}})$ is a metric such as

$$\rho(\underline{x},\hat{\underline{x}}) = \|\underline{x} - \hat{\underline{x}}\|^2 \tag{A.2-26}$$

For a specified source, the distribution of messages specified by $p(\underline{x})$ is fixed. By adjusting the encoder-channel-decoder combination, the conditional probability density function $p(\hat{\underline{x}}|\underline{x})$ can be manipulated. Each choice of $p(\hat{\underline{x}}|\underline{x})$ determines an information transfer rate, measured by $I(\underline{x};\hat{\underline{x}})$, and a distortion measured by $E[\rho(\underline{x},\hat{\underline{x}})]$. This corresponds to a point in the rate-distortion plane. See Fig. A.2-1. The lower right boundary of the region of possible performance combinations is called the rate-distortion curve (or rate distortion function). The rate-distortion curve for the source is determined by varying $p(\hat{\underline{x}}|\underline{x})$ to minimize $I(\underline{x},\hat{\underline{x}})$ subject to a distortion constraint

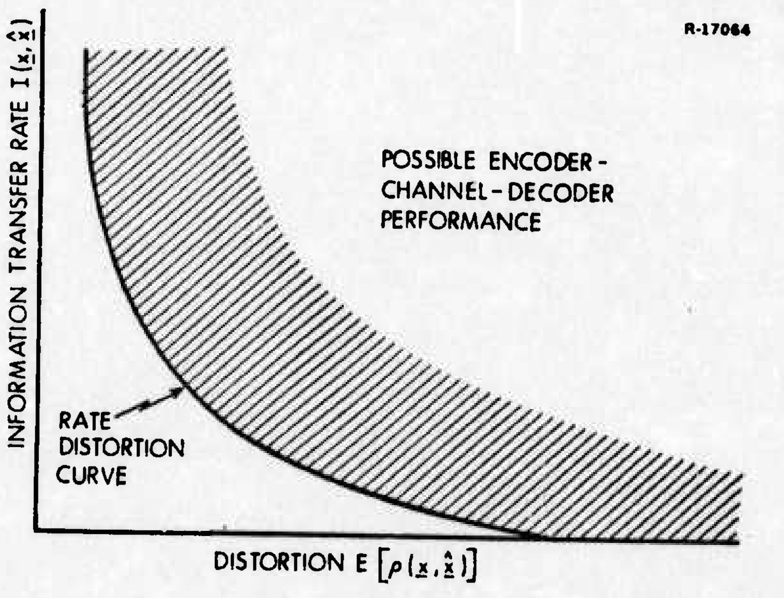

Figure A.2-1 Rate-Distortion Performance

$$E[\rho(\underline{x},\hat{\underline{x}})] = d \qquad \text{(A.2-27)}$$

Thus the rate-distortion curve specifies the minimum channel capacity required to maintain a specified distortion. Conversely, the rate-distortion curve also specifies the minimum distortion which can be achieved with a fixed channel capacity.

The rate-distortion function is defined by a constrained minimization problem which can be solved using Lagrange multiplier and calculus of variations methods. In Refs. 8 through 10, it is shown that the minimization produces

$$p(\underline{x}|\hat{\underline{x}}) = \beta(\underline{x})\; p(\underline{x})\; e^{s\rho(\underline{x},\hat{\underline{x}})} \qquad \text{(A.2-28)}$$

where $\beta(\underline{x})$ and s act as Lagrange multipliers. Here $\beta(\underline{x})$ is chosen to make

$$\int_{-\infty}^{\infty} p(\hat{\underline{x}}|\underline{x})\; d\hat{\underline{x}} = 1$$

for every $\underline{x}$, and s is chosen to produce a distortion of

$$E[\rho(\underline{x},\hat{\underline{x}})] = d$$

The minimum rate corresponding to this distortion is the resulting average mutual information, which is

$$R(d) = I(x;\hat{x}) = sd + \int_{-\infty}^{\infty} p(\underline{x})\; \ell n\; \beta(\underline{x})\; d\underline{x} \qquad \text{(A.2-29)}$$

An additional constraint is imposed implicitly because the average mutual information can never be negative. Therefore, for any compatible matrix A, the mutual information between $A\underline{x}$ and $A\hat{\underline{x}}$ must be non-negative.

$$E[\rho(\underline{x},\hat{\underline{x}})] = d \tag{A.2-27}$$

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$$\int_{-\infty}^{\infty} p(\hat{\underline{x}}|\underline{x})\ d\hat{\underline{x}} = 1$$

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An additional constraint is imposed implicitly because the average mutual information can never be negative. Therefore, for any compatible matrix A, the mutual information between $A\underline{x}$ and $A\hat{\underline{x}}$ must be non-negative.

If the distortion measure depends only on the difference between $\underline{x}$ and $\hat{\underline{x}}$, then it can be written as $\rho(\underline{x} - \hat{\underline{x}})$ and

$$p(\underline{x}|\hat{\underline{x}}) = \beta(\underline{x})\ p(\underline{x})\ e^{s\rho(\underline{x} - \hat{\underline{x}})} \tag{A.2-30}$$

Now for every $\hat{\underline{x}}$, this must be a probability density function. So

$$\int_{-\infty}^{\infty} p(\underline{x}|\hat{\underline{x}})\ d\underline{x} = 1 \tag{A.3-31}$$

for all $\hat{\underline{x}}$. If $\rho(\underline{x} - \hat{\underline{x}})$ is nonzero for all nonzero values of $(\underline{x} - \hat{\underline{x}})$, this implies that

$$p(\underline{x})\ \beta(\underline{x}) = \alpha \tag{A.3-32}$$

must be a constant.

A.3 RATE-DISTORTION CURVE FOR A GAUSSIAN VECTOR WITH QUADRATIC LOSS

An important special case results when the source distribution is Gaussian with mean zero and covariance matrix P^* and the distortion measure is the quadratic form

$$\begin{aligned} \rho(\underline{x} - \hat{\underline{x}}) &= \| W(\underline{x} - \hat{\underline{x}})\|^2 \\ &= (\underline{x} - \hat{\underline{x}})^T\ W^T\ W(\underline{x} - \hat{\underline{x}}) \end{aligned} \tag{A.3-1}$$

where W is a weighting matrix. It will be shown here, that the encoder-channel-decoder combination which minimizes the required channel capacity with a specified distortion can be acheived by first taking a measurement

$$\underline{z}^* = H^*\ \underline{x} + \underline{v}^* \tag{A.3-2}$$

and then setting

$$\hat{\underline{x}} = K^* \underline{z}^* \tag{A.3-3}$$

where

$\underline{v}^*$ is a zero mean Gaussian noise with covariance matrix R^*

H^* is the measurement matrix (which will be specified later in this section)

and

$$K^* = P^* H^{*T} \left[H^* P^* H^{*T} + R^* \right]^{-1} \tag{A.3-4}$$

To see this, first define

$$\underline{y} = D\, W\, \underline{x} \tag{A.3-5}$$

where D is an orthogonal matrix which diagonalizes $W\, P^*\, W^T$ to produce the matrix of eigenvalues

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \tag{A.3-6}$$

Now $\underline{x}$ can be expressed as

$$\underline{x} = \underline{a} + M\, \underline{y} \tag{A.3-7}$$

where $\underline{a}$ and $\underline{y}$ are independent. (This is a result of the projection theorem.) The estimate can also be written as

$$\hat{\underline{x}} = \hat{\underline{a}} + M\, \hat{\underline{y}} \tag{A.3-8}$$

where $\hat{a}$ and $\hat{y}$ are independent.

The average mutual information between $\underline{x}$ and $\hat{\underline{x}}$ is

$$I(\underline{x};\hat{\underline{x}}) = I(\underline{a};\hat{\underline{a}}) + I(\underline{y};\hat{\underline{y}}) \tag{A.3-9}$$

The distortion measure depends only on $(y - \hat{y})$, i.e.,

$$\rho(\underline{x} - \hat{\underline{x}}) = (\underline{x} - \hat{\underline{x}})^T W^T D^T D W(\underline{x} - \hat{\underline{x}})$$
$$= (\underline{y} - \hat{\underline{y}})^T (\underline{y} - \hat{\underline{y}}) \qquad \text{(A.3-10)}$$

Therefore, the average mutual information can be minimized by setting $\hat{a} = 0$ so that

$$I(\underline{a};\hat{\underline{a}}) = 0 \qquad \text{(A.3-11)}$$

and by then minimizing $I(\underline{y};\hat{\underline{y}})$. Since $\rho(\underline{y} - \hat{\underline{y}})$ is nonzero for all nonzero values of $(y - \hat{y})$, minimizing the average mutual information between y and $\hat{y}$ produces

$$p(\underline{y}|\hat{\underline{y}}) = \alpha \exp s(\underline{y} - \hat{\underline{y}})^T (\underline{y} - \hat{\underline{y}}) \qquad \text{(A.3-12)}$$

where α is a normalizing constant chosen to make

$$\int_{-\infty}^{\infty} p(\underline{y}|\hat{\underline{y}})\, dy = 1 \qquad \text{(A.3-13)}$$

Thus $\underline{y}$ given $\hat{\underline{y}}$ has a normal distribution with covariance matrix $-\frac{1}{2s}$ I. Requiring a distortion equal to d gives

$$\begin{aligned} E[\rho(\underline{x},\hat{\underline{x}})] &= \text{tr}\, E\left[(\underline{y} - \hat{\underline{y}})(\underline{y} - \hat{\underline{y}})^T\right] \\ &= \text{tr}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\underline{y}-\hat{\underline{y}})(\underline{y}-\hat{\underline{y}})^T \alpha \exp\left[-\tfrac{1}{2}(\underline{y}-\hat{\underline{y}})^T(-2s\, I)(\underline{y}-\hat{\underline{y}})\right] d\underline{y}\, p(\hat{y})\, d\hat{\underline{y}}\right\} \\ &= \text{tr}\left\{-\frac{1}{2s} I \int_{-\infty}^{\infty} p(\hat{\underline{y}})\, d\hat{\underline{y}}\right\} \\ &= -\frac{n}{2s} = d \end{aligned} \qquad \text{(A.3-14)}$$

where n is the dimension of $\underline{y}$. Therefore,

$$s = -\frac{n}{2d} \qquad \text{(A.3-15)}$$

and $\underline{y}$ given $\hat{\underline{y}}$ has the covariance matrix $\frac{d}{n}$ I.

The rate-distortion curve is given by the relationship

$$
\begin{aligned}
r^{o} &= I(\underline{y},\hat{\underline{y}}) = H(\underline{y}) - H(\underline{y}|\hat{\underline{y}}) \\
&= \frac{1}{2} \log \det \Lambda - \frac{1}{2} \log \det \frac{d^{o}}{n} I \qquad \text{(A.3-16)} \\
&= \frac{1}{2} \sum_{i=1}^{n} \left[\log \lambda_i - \log \frac{d^{o}}{n} \right]
\end{aligned}
$$

This rate applies if a negative mutual information does not occur in any direction A $\underline{y}$. Since the components of $\underline{y}$ are independent Gaussian variables, the requirement is

$$\log \lambda_i - \log \frac{d^{o}}{n} \geq 0 \quad \text{for every } i \qquad \text{(A.3-17)}$$

or

$$d^{o} \leq n\lambda_i \quad \text{for every } i \qquad \text{(A.3-18)}$$

When $n\lambda_i < d^{o}$ for some i, then no new information is required to estimate that component; the corresponding component of the unconditional mean is used. Suppose that the eigenvalues of $W\ P^{*}\ W^{T}$ are arranged in descending order of magnitude. As long as $d^{o} \leq n\lambda_n$, then

$$r^{o} = \sum_{i=1}^{n} \left[\log \lambda_i - \log \frac{d^{o}}{n} \right] \qquad \text{(A.3-19)}$$

For $d^{o} > n\lambda_n$, the problem becomes an n-1 dimensional problem with distortion constrained to be $d^{o} - \lambda_n$. Thus

$$r^{o} = \frac{1}{2} \sum_{i=1}^{n-1} \left[\log \lambda_i - \log \frac{d^{o}-\lambda_n}{n-1} \right] \qquad \text{(A.3-20)}$$

provided

$$\log \lambda_i - \log \frac{d^o - \lambda_n}{n-1} \geq 0 \quad \text{for } 1 \leq i \leq n-1$$

(A.3-21)

or

$$(n-1)\,\lambda_{n-1} + \lambda_n \geq d^o \tag{A.3-22}$$

In general for $(n-k)\lambda_{n-k} + \sum_{i=k+1}^{n} \lambda_i < d^o \leq (n-k+1)\lambda_{n-k+1} + \sum_{i=k+2}^{n} \lambda_i$

$$r^o = \frac{1}{2} \sum_{i=1}^{n-k} \left[\log \lambda_i - \log e\right] \tag{A.3-23}$$

where

$$e = \frac{d^o - \sum_{i=k+1}^{n} \lambda_i}{n-k} \tag{A.3-24}$$

In each case, the estimate $\hat{\underline{y}}$ can be viewed as the result of taking the (n-k)-dimensional measurement

$$\underline{z} = \Delta \underline{y} + \underline{v} \tag{A.3-25}$$

where $\underline{v}$ is a zero mean Gaussian noise with identity covariance matrix,

$$\Delta = \left[\begin{array}{cccc|c} \delta_1 & & 0 & & \\ & \delta_2 & & & \\ & & \ddots & & 0 \\ 0 & & & \delta_{n-k} & \end{array}\right] \tag{A.3-26}$$

and each δ_i is chosen to make the i^{th} component of the covarience of $\underline{y}$ given $\hat{\underline{y}}$ equal to e.

Using one form of the Kalman filter covariance equation (See Ref. 17, p.111) gives

$$\mathrm{Cov}\left[\underline{y}\,|\,\hat{\underline{y}}\right]^{-1} = \mathrm{Cov}[\underline{y}]^{-1} + \Delta^T \Delta \qquad \text{(A.3-27)}$$

which implies that

$$e^{-1} = \lambda_i^{-1} + \delta_i^2 \qquad \text{(A.3-28)}$$

or

$$\delta_i = \sqrt{\frac{\lambda_i - e}{e\ \lambda_i}} \qquad \text{(A.3-29)}$$

The measurement vector can also be written as

$$\underline{z}^* = H^* \underline{x} + \underline{v}^* \qquad \text{(A.3-20)}$$

where $\underline{v}^*$ has covariance matrix R^* and

$$H^* = R^{*\frac{1}{2}} \Delta D W \qquad \text{(A.3-31)}$$

($R^{*\frac{1}{2}}$ represents the symmetric square root of R^*.) For the purpose of determining the rate-distortion function, this measurement completely describes the optimum encoder-channel-decoder combination and the estimate $\hat{\underline{x}}$ is formed by using the Kalman filter equations.

A.4 RATE-DISTORTION CURVE FOR A GAUSS-MARKOV PROCESS WITH QUADRATIC LOSS

Suppose the source is described by a linear differential equation

$$\dot{\underline{x}}(t) = F(t)\ \underline{x}(t) + G(t)\ \underline{w}(t) \qquad \text{(A.4-1)}$$

where $\underline{w}$ is a white Gaussian noise with spectral matrix Q(t).

Assume that sufficient information has been gathered prior to time t_k to give a Gaussian conditional distribution for $\underline{x}(t_k)$ with covariance $P(t_k)$ and mean $\hat{\underline{x}}(t_k)$. Then $\underline{x}(t_{k+1})$ can be predicted by

$$\hat{\underline{x}}^*(t_{k+1}) = \Phi(t_{k+1})\,\hat{\underline{x}}(t_k) \tag{A.4-2}$$

The error in this prediction

$$\tilde{\underline{x}}^*(t_{k+1}) = \underline{x}(t_{k+1}) - \hat{\underline{x}}^*(t_{k+1}) \tag{A.4-3}$$

has a Gaussian distribution with covariance matrix

$$\begin{aligned} P^*(t_{k+1}) = {} & \Phi(t_{k+1},t_k)\, P(t_k)\, \Phi^T(t_{k+1},t_k) \\ & + Q^*(t_{k+1},t_k) \end{aligned} \tag{A.4-4}$$

where

$$Q^*(t_{k+1},t_k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},t)\, G(t)\, Q(t)\, G^T(g)\, \Phi^T(t_{k+1},t)\, dt \tag{A.4-5}$$

Assume that the distortion measure is the same as in the previous section. Note that

$$\underline{x}(t_{k+1}) = \tilde{\underline{x}}^*(t_{k+1}) + \hat{\underline{x}}^*(t_{k+1}) \tag{A.4-6}$$

so

$$\begin{aligned} \underline{x}(t_{k+1}) - \hat{\underline{x}}(t_{k+1}) = {} & \tilde{\underline{x}}^*(t_{k+1}) \\ & - \left[\hat{\underline{x}}(t_{k+1}) - \hat{\underline{x}}^*(t_{k+1})\right] \end{aligned} \tag{A.4-7}$$

Therefore the problem is one of estimating the Gaussian vector $\tilde{x}^*(t_{k+1})$ with the vector $\left[\hat{\underline{x}}(t_{k+1}) - \hat{\underline{x}}^*(t_{k+1})\right]$. That is,

the problem is the same as in the previous section with $\underline{\tilde{x}}^*(t_{k+1})$ taking the role of $\underline{x}$ and $\underline{\hat{x}}(t_{K+1}) - \underline{\hat{x}}^*(t_{k+1})$ taking the role of $\underline{\hat{x}}$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $P^*(t_{k+1})$ arranged in descending order. Then the rate-distortion curve is given by

$$r^o = \frac{1}{2} \sum_{i=1}^{n-k} \left[\log \lambda_i - \log \frac{d^o - \sum_{i=k+1}^{n} \lambda_i}{n-k} \right] \tag{A.4-8}$$

for $(n-k)\lambda_{n-k} + \sum_{i=k+1}^{n} \lambda_n < d^o \leq (n-k+1)\lambda_{n-k+1} + \sum_{i=k+2}^{n} \lambda_i$

A.5 RATE-DISTORTION CURVE FOR A LINEAR QUADRATIC GAUSSIAN PROBLEM WITH SENSOR CONSTRAINTS

In the previous section, no constraints were placed on the structure of the estimating system which forms $\underline{\hat{x}}(t_{k+1})$. However, in most practical problems, the estimating system is constrained to observe only measurements of the form

$$\underline{z}(t) = H(t)\, \underline{x}(t) + \underline{v}(t) \tag{A.5-1}$$

either continuously or discretely in time. Here $\underline{v}(t)$ is a white measurement noise. Results of Wolf and Ziv (Ref. 4) and Dobrushin and Tsybakov (Ref. 5) can be extended to show that the optimal encoder can be partitioned into a Kalman filter which preprocesses the measurements followed by an encoder which treats the Kalman filter as its source as shown in Fig. A.5-1 (see Ref. 13).

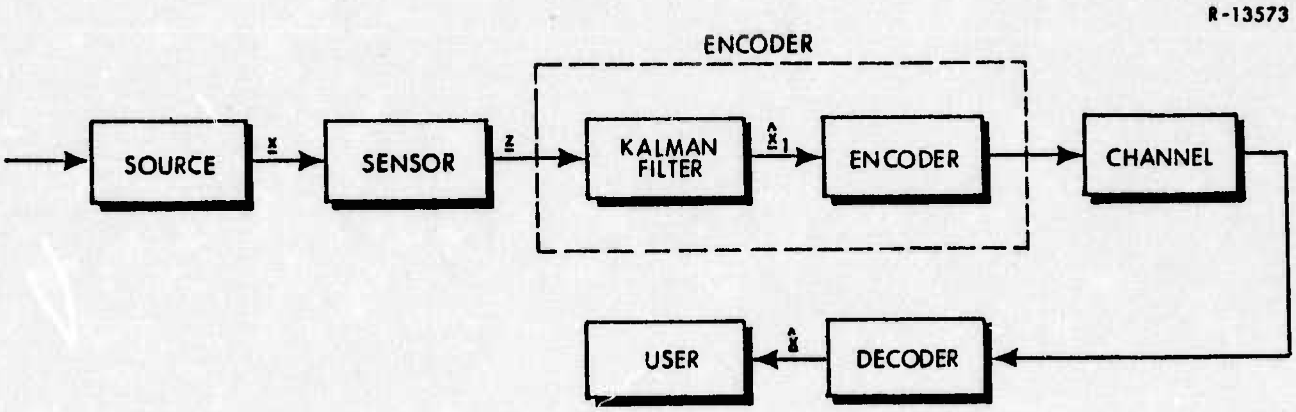

Figure A.5-1   Optimal Encoder for System with Sensor Constraints

If the measurements are available continuously, the Kalman filter is a continuous time linear system driven by the innovations process $\underline{\nu}(t)$,

$$\dot{\hat{\underline{x}}}_1(t) = F(t)\,\hat{\underline{x}}_1(t) + K(t)\,\underline{\nu}(t) \qquad \text{(A.5-2)}$$

The innovations process is a Gaussian white noise process with covariance matrix R(t), the same covariance as the measurement noise.

If the measurements are available only at discrete instants of time, the Kalman filter is a discrete time linear system driven by an innovations sequence.

$$\begin{aligned}\hat{\underline{x}}_1(t_{k+1}) &= \Phi(t_{k+1},t_k)\,\hat{\underline{x}}_1(t_k) \\ &\quad + K(t_{k+1})\,\underline{\nu}(t_{k+1})\end{aligned} \qquad \text{(A.5-3)}$$

The covariance of the innovations is

$$\begin{aligned}N(t_{k+1}) &= H(t_{k+1})\,P_1^*(t_{k+1})\,H^T(t_{k+1}) \\ &\quad + R(t_{k+1})\end{aligned} \qquad \text{(A.5-4)}$$

where $P_1^*(t_{k+1})$ is the covariance matrix for the Kalman prefilter's error in predicting $\underline{x}(t_{k+1})$.

In either of these cases, the techniques developed in the previous section can be used to compute the rate distortion function. The decoder now constructs an estimate $\hat{\underline{x}}(t_{k+1})$ for the state of the Kalman prefilter $\hat{\underline{x}}_1(t_{k+1})$. Assume that sufficient data has been gathered prior to t_k to produce a Gaussian distribution for $\hat{\underline{x}}_1(t_k)$ with covariance matrix $P(t_k)$. The covariance matrix for the error the receiver makes in predicting $\hat{\underline{x}}_1(t_{k+1})$ based on this data is

$$P^*(t_{k+1}) = \Phi(t_{k+1},t_k)\, P(t_k)\, \Phi^T(t_{k+1},t_k) + N^*(t_{k+1},t_k) \tag{A.5-5}$$

where

$$N^*(t_{k+1},t_k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},t)\, K(t)\, R(t)\, K^T(t)\, \Phi^T(t_{k+1},t)\, dt \tag{A.5-6}$$

for continuous measurements and

$$N^*(t_{k+1},t_k) = \sum_{t_k<t \leq t_{k+1}} \Phi(t_{k+1},t)\, K(t)\, N(t)\, K^T(t)\, \Phi^T(t_{k+1},t)\, dt \tag{A.5-7}$$

for discrete measurements. The eigenvalues of $P^*(t_{k+1})$ are used to determine the rate distortion function.

There is one additional alteration to the previous techniques. The distortion consists of

$$E[\rho(\underline{x},\hat{\underline{x}})] = \text{tr } E\Big\{W\big[\underline{x}(t_{k+1}) - \hat{\underline{x}}_1(t_{k+1})\big]\big[\underline{x}(t_{k+1}) - \hat{\underline{x}}_1(t_{k+1})\big]^T W^T$$
$$+ W\big[\hat{\underline{x}}_1(t_{k+1}) - \hat{\underline{x}}(t_{k+1})\big]\big[\hat{\underline{x}}(t_{k+1}) - \hat{\underline{x}}(t_{k+1})\big]W^T\Big\}$$
$$= W\, P_1(t_{k+1})\, W^T + W\, P(t_{k+1})\, W^T$$

(A.5-8)

Therefore the distortion cannot be made smaller than

$$d_o = W\, P_1(t_{k+1})\, W^T \tag{A.5-9}$$

and the rate-distortion function is

$$r^o = \frac{1}{2}\sum_{i=1}^{n-k}\left[\log \lambda_i - \log \frac{d^o - d_o - \sum\limits_{i=k+1}^{n} \lambda_i}{n-k}\right]$$

(A.5-10)

for

$$(n-k)\lambda_{n-k} + \sum_{i=k+1}^{n} \lambda_i + d_o < d^o \le (n-k+1)\lambda_{n-k+1} + \sum_{i=k+2}^{n} \lambda_i + d_o$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $W\, P^*(t_{k+1})\, W^T$ arranged in descending order of magnitude.

APPENDIX B

DERIVATION OF THE ERROR COVARIANCE EQUATIONS

The equations in Chapter 4 can be used to produce a recursion relation for the one-step prediction error. First note that

$$\begin{aligned}
\tilde{\underline{x}}_{n+1|n} &= \underline{x}_{n+1} - C_{n+1}\,\underline{m}_{n+1} \\
&= \Phi_n\,\underline{x}_n + \underline{w}_n - C_{n+1}\,A_n\,\underline{m}_n \\
&\quad - C_{n+1}\,B_n(H_n\,\underline{x}_n + \underline{v}_n - G_n\,\underline{m}_n) \\
&= (\Phi_n - C_{n+1}\,B_n\,H_n)\,\underline{x}_n - C_{n+1}(A_n - B_n\,G_n)\,\underline{m}_n \\
&\quad - \underline{w}_n - C_{n+1}\,B_n\,\underline{v}_n
\end{aligned} \tag{B-1}$$

Recognizing that

$$\underline{x}_n = \tilde{\underline{x}}_{n|n-1} + C_n\,\underline{m}_n \tag{B-2}$$

gives

$$\boxed{\begin{aligned}
\tilde{\underline{x}}_{n+1|n} &= (\Phi_n - C_{n+1}\,B_n\,H_n)\,\tilde{\underline{x}}_{n|n-1} \\
&\quad + \Big[(\Phi_n - C_{n+1}\,B_n\,H_n)C_n - C_{n+1}(A_n - B_n\,G_n)\Big]\,\underline{m}_n \\
&\quad + \underline{w}_n - C_{n+1}\,B_n\,\underline{v}_n
\end{aligned}}$$

and (B-3)

$$\underline{m}_{n+1} = A_n \underline{m}_n + B_n(H_n \underline{x}_n + \underline{v}_n - G_n \underline{m}_n)$$

$$= (A_n - B_n G_n) \underline{m}_n + B_n H_n \underline{x}_n + B_n \underline{v}_n$$

$$\boxed{\underline{m}_{n+1} = \left[A_n + B_n(H_n C_n - G_n)\right] \underline{m}_n + B_n H_n \tilde{\underline{x}}_{n|n-1} + B_n \underline{v}_n} \qquad \text{(B-4)}$$

Therefore, $\tilde{\underline{x}}_{n+1|n}$ and $\underline{m}_{n+1}$ satisfy the coupled set of linear vector equations

$$\begin{bmatrix} \tilde{\underline{x}}_{n+1|n} \\ \underline{m}_{n+1} \end{bmatrix} = \begin{bmatrix} \Delta_{11}(n) & \Delta_{12}(n) \\ \Delta_{21}(n) & \Delta_{22}(n) \end{bmatrix} \begin{bmatrix} \tilde{\underline{x}}_{n|n-1} \\ \underline{m}_n \end{bmatrix} + \begin{bmatrix} \Lambda_1(n) \\ \Lambda_2(n) \end{bmatrix} \underline{v}_n + \begin{bmatrix} I \\ 0 \end{bmatrix} \underline{w}_n \qquad \text{(B-5)}$$

where

$$\Delta_{11}(n) = \Phi_n - C_{n+1} B_n H_n \qquad \text{(B-6)}$$

$$\Delta_{12}(n) = (\Phi_n - C_{n+1} B_n H_n) C_n - C_{n+1}(A_n - B_n G_n) \qquad \text{(B-7)}$$

$$\Delta_{21}(n) = B_n H_n \qquad \text{(B-8)}$$

$$\Delta_{22}(n) = A_n + B_n(H_n C_n - G_n) \qquad \text{(B-9)}$$

$$\Lambda_1(n) = - C_{n+1} B_n \qquad \text{(B-10)}$$

and

$$\Lambda_2(n) = B_n \qquad \text{(B-11)}$$

Define

$$\Pi_n = E\begin{bmatrix} \underline{\tilde{x}}_{n|n-1}\ \underline{\tilde{x}}^T_{n|n-1} & \underline{\tilde{x}}_{n|n-1}\ \underline{m}^T_n \\ \underline{m}_n\ \underline{\tilde{x}}^T_{n|n-1} & \underline{m}_n\ \underline{m}^T_n \end{bmatrix} \tag{B-12}$$

and the previous equation gives

$$\Pi_{n+1} = \Delta_n\ \Pi_n\ \Delta^T_n + \Lambda_n\ R_n\ \Lambda^T_n + \begin{bmatrix} Q_n & 0 \\ 0 & 0 \end{bmatrix} \tag{B-13}$$

This recursion relation is started with

$$\Pi_o = \begin{bmatrix} P^*_o + (I-C_o\ T_o)\ X_o(I-C_o\ T_o)^T & (I-C_o\ T_o)\ X_o\ T^T_o \\ T_o\ X_o(I-C_o\ T_o)^T & T_o\ X_o\ T^T_o \end{bmatrix} \tag{B-14}$$

where

$$X_o = E[\underline{x}_o]\ E[\underline{x}_o]^T \tag{B-15}$$

$$P^*_o = E\left[(\underline{x} - E[\underline{x}_o])(\underline{x} - E[\underline{x}_o])^T\right] \tag{B-16}$$

m_o is chosen to be $T_o\ E[\underline{x}_o]$

and

$$\begin{aligned} C_o &= E\left[\underline{x}_o\ \underline{m}^T_o\right]\ E\left[\underline{m}_o\ \underline{m}^T_o\right]^{-1} \\ &= X_o\ T^T_o\ \left[T_o\ X_o\ T^T_o\right]^{-1} \end{aligned} \tag{B-17}$$

The state estimation error is defined to be

$$\underline{\tilde{x}}_{n|n} = \underline{x}_n - \underline{\hat{x}}_{n|n} \tag{B-18}$$

which can be expanded to

$$\underline{\tilde{x}}_{n|n} = \underline{x}_n - C_n \underline{m}_n - K_n(H_n \underline{x}_n + \underline{v}_n - G_n \underline{m}_n)$$
$$= (I - K_n H_n) \underline{x}_n - (C_n - K_n G_n) \underline{m}_n - K_n \underline{v}_n \tag{B-19}$$

Recalling that $\underline{x}_n = \underline{\tilde{x}}_{n|n-1} + C_n \underline{m}_n$ gives

$$\underline{\tilde{x}}_{n|n} = (I-K_n H_n) \underline{\tilde{x}}_{n|n-1} + \Big[(I-K_n H_n) C_n - (C_n - K_n G_n)\Big]\underline{m}_n - K_n \underline{v}_n$$
$$= (I-K_n H_n) \underline{\tilde{x}}_{n|n-1} - K_n(G_n - H_n C_n) \underline{m}_n - K_n \underline{v}_n$$
$$= \begin{bmatrix}\Psi_1(n) & \Psi_2(n)\end{bmatrix} \begin{bmatrix} \underline{\tilde{x}}_{n|n-1} \\ \underline{m}_n \end{bmatrix} - K_n \underline{v}_n \tag{B-20}$$

Then the estimation error covariance is

$$\boxed{\begin{aligned} P_n &= E\Big[\underline{\tilde{x}}_{n|n} \underline{\tilde{x}}_{n|n}^T\Big] \\ &= \Psi_n \Pi_n \Psi_n + K_n R_n K_n^T \end{aligned}} \tag{B-21}$$

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